SOE-RBC: Quantitative Analysis

Based on *OEM* chapter 4

November 4, 2020

Motivation

- So far we've built a model of the open economy driven by productivity shocks and argued that it can capture the observed countercyclicality of the trade balance.

- We also established that two features of the model are important for making this prediction possible:
  1. productivity shocks must be sufficiently persistent.
  2. capital adjustment costs must not be too strong.

- **Now**, we ask more questions about the ability of that model to explain observed business cycles. Can it explain ...
  - the sign and magnitude of business-cycle indicators, such as the standard deviation, serial correlation, and correlation with output of output, consumption, investment, the trade balance, and the current account.

To make the models studied in chapters 2 and 3 more empirically realistic and to give them a better chance to account for observed business-cycle regularities add:

1. endogenous labor supply and demand
2. uncertainty in the technology shock process
3. capital depreciation.

The resulting theoretical framework is known as the Small Open Economy Real-Business-Cycle model, or, succinctly, the SOE-RBC model.

The Household's Maximization Problem

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, h_t) \tag{4.1}$$

subject to

$$c_t + i_t + \Phi(k_{t+1} - k_t) + (1 + r_{t-1})d_{t-1} = y_t + d_t \tag{4.2}$$

$$y_t = A_t F(k_t, h_t) \tag{4.3}$$

$$k_{t+1} = (1 - \delta)k_t + i_t \tag{4.4}$$

$$\lim_{j \to \infty} E_t \frac{d_{t+j}}{\prod_{s=0}^{j}(1 + r_s)} \leq 0 \tag{4.5}$$

Capital adjustment cost, $\Phi(0) = \Phi'(0) = 0$; $\Phi''(0) > 0$

The Household's Maximization Problem

Additions/differences to the model analyzed in Chapter 3

- endogenous labor supply, $U(c_t, h_t)$
- endogenous labor demand, $F(k_t, h_t)$
- uncertainty, A_t is stochastic
- the interest rate is no longer constant, $r_t \neq r$
- depreciation, δ no longer 0

Household's Optimality Conditions

$$c_t + k_{t+1} - (1-\delta)k_t + \Phi(k_{t+1} - k_t) + (1 + r_{t-1})d_{t-1} = A_t F(k_t, h_t) + d_t \tag{4.6}$$

$$\lambda_t = \beta(1 + r_t)E_t\lambda_{t+1} \tag{4.7}$$

$$U_c(c_t, h_t) = \lambda_t \tag{4.8}$$

$$-U_h(c_t, h_t) = \lambda_t A_t F_h(k_t, h_t) \tag{4.9}$$

$$1 + \Phi'(k_{t+1} - k_t) = \beta E_t \frac{\lambda_{t+1}}{\lambda_t}\left[A_{t+1}F_k(k_{t+1}, h_{t+1}) + 1 - \delta + \Phi'(k_{t+2} - k_{t+1})\right] \tag{4.10}$$

- Let's take a quick stop here and let's think about the debt-Euler equation again:

$$U_c(c_t, h_t) = \beta(1 + r_t)E_t U_c(c_{t+1}, h_{t+1})$$

What happens if we impose steady state?

Inducing Stationarity: External debt-Elastic Interest Rate (EDEIR)

$$r_t = r^* + p(\widetilde{d}_t) \tag{4.14}$$

r^* = constant world interest rate
$p(\tilde{d}_t)$ = country interest-rate premium
$\tilde{d}_t$ = cross-sectional average of debt

In equilibrium cross-sectional average of debt must equal individual debt

$$\widetilde{d}_t = d_t \tag{4.15}$$

Evolution of Total Factor Productivity, AR(1) process

$$\ln A_{t+1} = \rho \ln A_t + \widetilde{\eta}\epsilon_{t+1} \tag{4.12}$$

The Trade Balance

$$tb_t = y_t - c_t - i_t - \Phi(k_{t+1} - k_t) \tag{4.20}$$

The Current Account

$$ca_t = tb_t - r_{t-1}d_{t-1} \tag{4.21}$$

Equilibrium Conditions

$$-\frac{U_h(c_t,h_t)}{U_c(c_t,h_t)} = A_t F_h(k_t,h_t) \tag{4.11}$$

$$c_t + k_{t+1} - (1-\delta)k_t + \Phi(k_{t+1}-k_t) + [1+r^* + p(d_{t-1})]d_{t-1} = A_t F(k_t,h_t) + d_t \tag{4.16}$$

$$U_c(c_t,h_t) = \beta(1+r^*+p(d_t))E_t U_c(c_{t+1},h_{t+1}) \tag{4.17}$$

$$1 = \beta E_t \left\{ \frac{U_c(c_{t+1},h_{t+1})}{U_c(c_t,h_t)} \frac{[A_{t+1}F_k(k_{t+1},h_{t+1}) + 1 - \delta + \Phi'(k_{t+2}-k_{t+1})]}{1+\Phi'(k_{t+1}-k_t)} \right\} \tag{4.18}$$

This is a system of non-linear stochastic difference equations. It does not have a closed form solution. We will use numerical techniques to find a first-order accurate approximate solution around the nonstochastic steady state. This is a local approximation. (Later we will also consider a global solution method.)

Equilibrium Conditions (cont'd)

Note:
For capital, the system is a second-order difference equation as it features k_t, k_{t+1} and k_{t+2}. We would like to have a system of first-order difference equations. To this end, introduce the auxiliary variable k_t^f and impose

$$k_t^f = k_{t+1}$$

Note that k_t^f is in the information set of period t. This equation together with the four above equations forms a system of stochastic first-order difference equations in the 5 unknowns: c_t, h_t, d_{t-1}, k_t, and k_t^f.

Now what?

- We've developed a RBC model of a SOE in the previous classes.
- Now what?
  - Under special assumptions: Analytic solutions (i.e. closed-form decision rules).
  - In general: we need to rely on numerical methods
- One of the most common approaches: first-order perturbation (or linearization) around the deterministic s.s.

- Q: How do we compute the "deterministic s.s." of our model?

- We first need to determine functional forms.

Functional Forms

- First-order pertubation method requires Taylor approximations around the deterministic s.s.
- In turn, this requires first-derivatives of eq. conditions $\rightarrow$ tedious process for large models $\rightarrow$ use of statistical software.
- Finally, this implies we need specific functional forms: preferences and technologies.

Preferences

$$U(c,h) = \frac{G(c,h)^{1-\sigma} - 1}{1-\sigma}$$

with $\sigma > 0$, and:

$$G(c,h) = c - \frac{h^{\omega}}{\omega}$$

where $\omega > 1$.

Functional Forms - GHH preferences

- These preferences are due to Greenwood, Hercowitz, and Huffman (1988 AER) and are typically called "GHH preferences".
- Neat property: MRS btw consumption and leisure depends only on labor.
    - Labor supply is income/wealth inelastic.
- Use (4.8), (4.9) and the GHH form:

$$h_t^{\omega-1} = w_t \quad (= A_t F_h(k_t, h_t))$$

- Wage elasticity of labor supply: $1/(\omega - 1)$ and **independent** from cons.

- $U(c, h)$ is CRRA over sub-utility index $G(c, h)$
- σ degree of RRA, and $1/\sigma$ IES.

Production Function

- Typically assume Cobb-Douglas prod. func.

$$F(k,h) = k^{\alpha} h^{1-\alpha}, \quad \alpha \in (0,1)$$

- Unitary elasticity of subst. btw capital and labor
- Devide (25) by (26):

$$\frac{1-\alpha}{\alpha} \frac{k_t}{h_t} = \frac{w_t}{u_t}$$

- ... take logs in both sides...

$$\log\left(\frac{k_t}{h_t}\right) = \log\left(\frac{w_t}{u_t}\right) - const$$

Capital-labor ratio is proportional to the wage-rental ratio.

Capital Adjustment Cost

$$\Phi(x) = \frac{\phi}{2}x^2$$

- $\phi > 0$. Net investment (+ or -) generates adj. costs.

EDEIR form

$$p(d) = \psi_1 \left(e^{d-\bar{d}} - 1\right)$$

- $\psi_1 > 0$, and $\bar{d}$ are parameters.
- Country premium is increasing and convex in net external debt.

Deterministic Steady State

- Recall stochastic process:

$$logA_{t+1} = \rho logA_t + \tilde{\eta}\epsilon_{t+1} \tag{12}$$

  where $\rho \in (-1, 1)$.
- A deterministic economy is one with $\tilde{\eta} = 0$.
- A deterministic s.s. is an equilibrium of that economy in which all endogenous variables are constant over time.

- Why do we care?
  1. Steady state facilitates calibration because it is $\approx$ average position of the stochastic economy.
  2. Det. s.s. often used as point of reference for linearization.

Deriving the Deterministic Steady State

Remember optimality conditions:

$$\frac{-U_h(c_t, h_t)}{U_c(c_t, h_t)} = A_t F_h(k_t, h_t) \tag{11}$$

$$\begin{aligned} d_t = \quad & (1 + r_{t-1})d_{t-1} - A_t F(k_t, h_t) + c_t + k_{t+1} - (1-\delta)k_t + ... \\ & \Phi(k_{t+1} - k_t) \end{aligned} \tag{15}$$

$$U_c(c_t, h_t) = \beta(1 + r^* + p(d_t))E_t U_c(c_{t+1}, h_{t+1}) \tag{16}$$

$$\begin{aligned} U_c(c_t, h_t) = \quad & \beta E_t \frac{U_c(c_{t+1}, h_{t+1})}{[1 + \Phi'(k_{t+1} - k_t)]}[A_{t+1} F_k(k_{t+1}, h_{t+1}) + ... \\ & 1 - \delta + \Phi'(k_{t+2} - k_{t+1})] \end{aligned} \tag{17}$$

- For any variable, we denote the s.s. value by removing “t” subscript.

Deriving the Deterministic Steady State - II

From (16) we obtain:

$$1 = \beta[1 + r^* + \psi_1(e^{d-\bar{d}} - 1)]$$

w.l.o.g. assume: $\beta(1 + r^*) = 1$. Combining we get: $d = \bar{d}$.

The s.s. version of (17) reads:

$$1 = \beta \left[\alpha \left(\frac{k}{h} \right)^{(\alpha-1)} + 1 - \delta \right]$$

From here we get the s.s. expression for the capital-labor ratio:

$$k/h = \left(\frac{\beta^{-1} - 1 + \delta}{\alpha} \right)^{1/(\alpha-1)} \equiv \kappa$$

Deriving the Deterministic Steady State - III

Using κ in the s.s. version of (11), we obtain:

$$h = [(1-\alpha)\kappa^{\alpha}]^{1/(\omega-1)}$$

Next, given the s.s. values for capital-labor ratio and labor, we get:

$$k = \kappa h$$

Finally, use (15) in s.s. to obtain:

$$c = -r^* \bar{d} + \kappa^{\alpha} h - \delta k$$

$\rightarrow$ We now have s.s. values for: c, h, k, d

Calibration

- There is an "art" in assigning values to structural parameters in our models.
- In general, two approaches: estimation and **calibration**.

- Calibration assigns values in three ways
  - (a) Based on sources unrelated to the macro variables we want to explain.
  - (b) To match first order moments of data we want to explain.
  - (c) To match second-order moments of data we want to explain.

- Calibration of SOE-RBC based on Mendoza (1991 AER). Data: Canada. Time unit: 1 year. Quadratic detrending.
- 10 parameters: $\sigma, \delta, r^*, \alpha, \bar{d}, \omega, \phi, \psi_1, \rho$ and $\tilde{\eta}$.

(a) Parameters based on sources unrelated to the macro variables we want to explain

Concept	Parameter	Value
RRA	σ	2
World Interest Rate	r^*	0.04
Depreciation rate	δ	0.1

Calibration - III

(b) Parameters set to match 1st-order moments of data we want to explain

- Two parameters: α and $\bar{d}$.
- $\alpha \rightarrow$ average labor share in Canada (0.68).
  In our model: $\frac{wh}{y} = 1 - \alpha \Rightarrow \alpha = 0.32$.
- $\bar{d} \rightarrow$ to match average tb/y in Canada (0.02).
  Using definition of TB (eq. 19) and (15):

$$tb = r^* \bar{d}$$

  In the det. s. s. we should run a TB large enough to just service external debt
- Solving for $\bar{d}$:

$$\bar{d} = \frac{tb/y}{r^*} y$$

(b) Parameters set to match 1st-order moments of data we want to explain (cont'd)

$$\bar{d} = \frac{tb/y}{r^*} y$$

- We know $tb/y = 0.02$ and $r = 0.04$, but don't know y.
- From s.s. derivation we can obtain:

$$y = [(1-\alpha)\kappa^{\alpha\omega}]^{1/(\omega-1)}$$

where $\kappa = \left(\frac{\alpha}{r^*+\delta}\right)^{1/(1-\alpha)}$

- The only unknown is ω.

(c) Parameters set to match 2nd-order moments of data we want to explain

- Five parameters: $\omega, \phi, \psi_1, \rho, \tilde{\eta}$.
- Moments to match: std. dev. of hours worked (2.02%), std. dev. investment (9.82%), std. dev. *tb*/*y* (1.87%), serial corr. of output (0.62), and std. dev. output (2.81%).
- Calibration Strategy:
  1. Guess five value
  2. Approximate the equilibrium dynamics
  3. Compute model-implied moments of interest
  4. If $\exists$ match btw data and model-generated moments, convergence achieved.
     If not, update guess and return to step (1).

Calibration - VI

Concept	Parameter	Value
IES	σ	2
World Interest Rate	r^*	0.04
Depreciation rate	δ	0.1
Labor Share	α	0.32
Avg. Debt level in s.s.	$\bar{d}$	0.7442
Frisch elasticity	ω	1.455
Adj. Cost param	ϕ	0.028
Int. rate debt elasticity	ψ_1	0.000742
TFP Shock persistence	ρ	0.42
TFP Shock std. dev.	$\tilde{\eta}$	0.0129

NOTE: This is just one possible calibration strategy.

App. equilibrium dynamics - SGU algorithm

- Numerical solution of SOE-RBC $\rightarrow$ multiple methods; here only **linear app.**
- Still, more than one option: Dynare, Uligh's Toolkit, KPR, etc. Here SGU (2004 JEDC).

- Some vars. expressed in deviations from s.s. $\rightarrow c_t, h_t, k_t, A_t$. Other vars expressed in levels $\rightarrow r_t, d_{t-1}$.
- Define $y_t \equiv [\ln c_t \ln h_t]'$, vector of control variables.
- Define $x_t^1 \equiv [d_{t-1} \ln k_t]'$, vector of endogenous state variables.
- Define $x_t^2 \equiv [\ln A_t]'$, vector of exogenous state variables.
  $\Rightarrow x_t \equiv [x_t^{1'} x_t^{2'}]'$ vector of state variables.
- Next step: Apply conditional expectations to (12) $\rightarrow E_t lnA_{t+1} - \rho lnA_t = 0$.
- Together with (11), (15)-(17) we can write:

$$E_t f(y_{t+1}, y_t, x_{t+1}, x_t) = 0 \tag{28}$$

App. equilibrium dynamics - SGU algorithm II

- The law of motion for x_t^2 is:

$$x_{t+1}^2 = \rho x_t^2 + \tilde{\eta}\epsilon_{t+1} \tag{29}$$

- Next, take care of TVC (eq. 18):

$$\lim_{j\to\infty} \begin{bmatrix} E_t y_{t+1} \\ E_t x_{t+j} \end{bmatrix} = \begin{bmatrix} y^{ss} \\ x^{ss} \end{bmatrix} \tag{30}$$

- The first-order accurate solution to (28)-(30) is given by:

$$\hat{x}_{t+1} = h_x \hat{x}_t + \eta\epsilon_{t+1}$$

and

$$\hat{y}_t = g_x \hat{x}_t$$

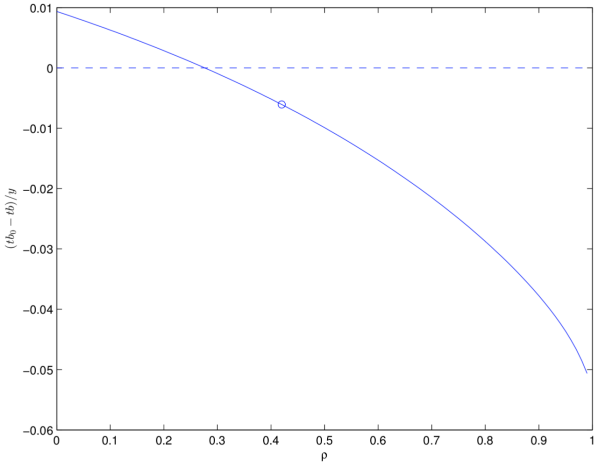
where $\hat{x}_t = x_t - x^{ss}$ and $\hat{y}_t = y_t - y^{ss}$.

Predictions of our model

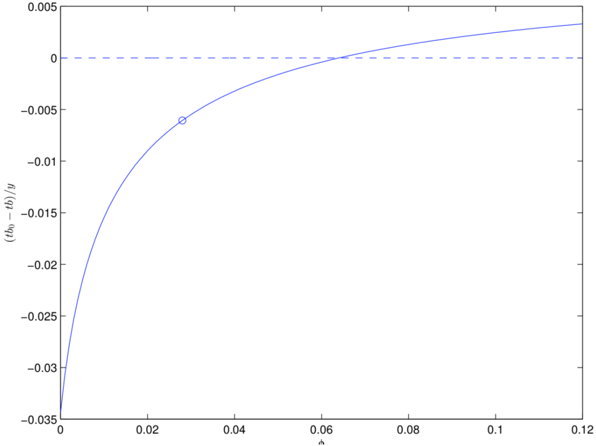
Before analyzing to which extend the SOE-RBC model can account for the observed Canadian business cycle, let's first study the predictions of this model building on the intuition from the simpler versions. In particular, we learned that:

- the more persistent productivity shocks are, the more likely an initial deterioration of the trade balance will be.
- the more pronounced are capital adjustment costs, the smaller will be the initial trade balance deterioration in response to a positive and persistent productivity shock.
- the more persistent the technology shock is, the higher the volatility of consumption relative to output will be. The next three figures show that these analytical results do indeed hold in the fully-fledged stochastic dynamic open economy real-business-cycle model.

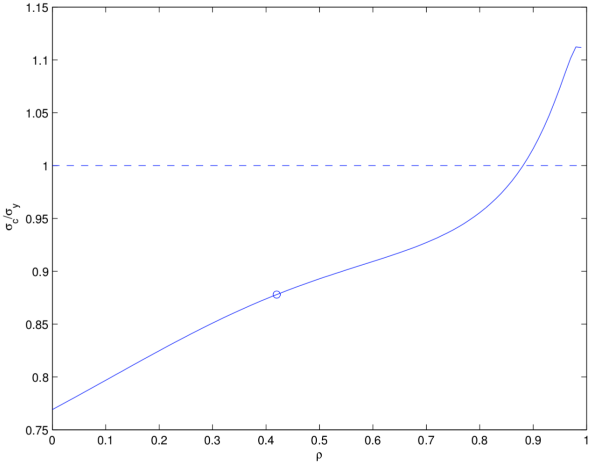
Impact on tb/y as a function of ρ



Impact on tb/y as a function of ϕ



σ_c/σ_y as a function of ρ



Empirical regularities of the Canadian Economy

Why Canada? Because it is a small open economy and it is the economy studied in Mendoza (1991).

Variable	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}
y	2.81	0.62	1
c	2.46	0.70	0.59
i	9.82	0.31	0.64
h	2.02	0.54	0.80
tb/y	1.87	0.66	-0.13

Performance of the model - 2nd moments

Variable	Canadian Data			Model		
	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}
y	2.81	0.62	1	3.08	0.62	1.00
c	2.46	0.70	0.59	2.71	0.78	0.84
i	9.82	0.31	0.64	9.04	0.07	0.67
h	2.02	0.54	0.80	2.12	0.62	1.00
tb/y	1.87	0.66	-0.13	1.78	0.51	-0.04
ca/y				1.45	0.32	0.05

Performance of the model - 2nd moments

Variable	Canadian Data			Model		
	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}
y	2.81	0.62	1	**3.08**	**0.62**	1.00
c	2.46	0.70	0.59	2.71	0.78	0.84
i	9.82	0.31	0.64	**9.04**	0.07	0.67
h	2.02	0.54	0.80	**2.12**	0.62	1.00
tb/y	1.87	0.66	-0.13	**1.78**	0.51	-0.04
ca/y				1.45	0.32	0.05

Performance of the model - 2nd moments

Variable	Canadian Data			Model		
	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}
y	2.81	0.62	1	**3.08**	**0.62**	1.00
c	2.46	0.70	0.59	**2.71**	0.78	0.84
i	9.82	0.31	0.64	**9.04**	0.07	0.67
h	2.02	0.54	0.80	**2.12**	0.62	1.00
tb/y	1.87	0.66	-0.13	**1.78**	0.51	**-0.04**
ca/y				1.45	0.32	0.05

Performance of the model - 2nd moments

Variable	Canadian Data			Model		
	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}
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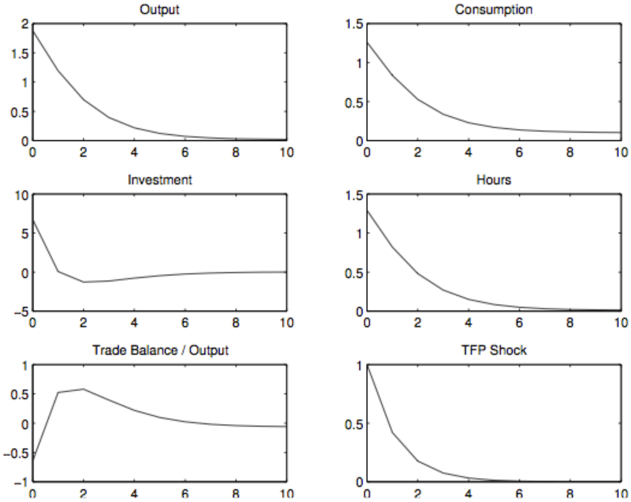
c	2.46	0.70	0.59	**2.71**	0.78	**0.84**
i	9.82	0.31	0.64	**9.04**	0.07	0.67
h	2.02	0.54	0.80	**2.12**	0.62	**1.00**
tb/y	1.87	0.66	-0.13	**1.78**	0.51	-**0.04**
ca/y				1.45	0.32	0.05

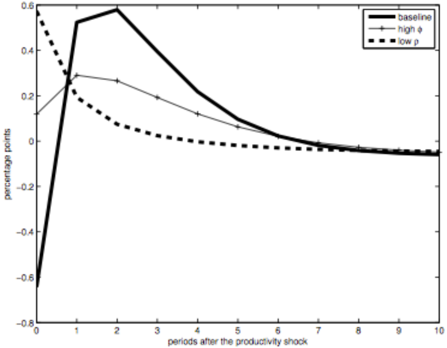
Performance of the model - IRFs

Figure 4.1: Responses to a One-Percent Productivity Shock



Performance of the model - Role of ρ and adj. costs on TB

Figure 4.2: Response of the Trade-Balance-To-Output Ratio to a Positive Technology Shock



Broken line: $\rho = 0.21$. Crossed line: $\phi = 0.084$.

Remember Principles I and II !

SOE-RBC with Complete Asset Mkts

Variable	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}
y	3.1	0.61	1.00
c	1.9	0.61	1.00
i	9.1	0.07	0.66
h	2.1	0.61	1.00
tb/y	1.6	0.39	0.13
ca/y	3.1	-0.07	-0.49

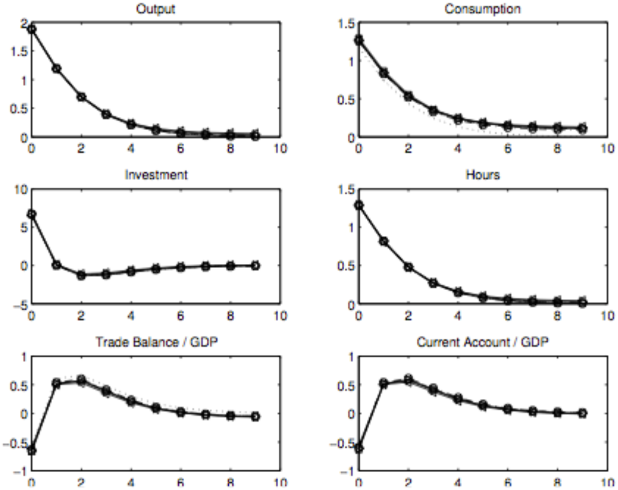
Other ways to induce stationarity

1. Internal Debt-Elastic Interest Rate Model (IDEIR)
2. Portfolio Adjustment Cost Model (PAC)
3. External Discount Factor Model (EDF) – Uzawa-Epstein-Zin preferences
4. Internal Discount Factor Model (EDF)
5. Perpetual Youth Model (PY)
6. *Complete Asset Markets* Model (CAM)

All of these work with linearized solution techniques. Moreover, the business cycle implications (when calibrated properly) are almost identical (SGU, JIE 2003).

IRFs across models

Figure 4.3: Impulse Response to a Unit Technology Shock Across Models



Global solution

- Everything so far has been assuming a linearized solution (around the deterministic s.s.)
- Alternatively, one can solve the model "globally".
- We can induce stationarity in this model by assuming

$$\beta(1 + r^*) < 1$$

- Solution algorithm based on VFI:

$$v(d, k, A) = \max_{d', k'} \Big\{ U(AF(k, h) + (1 - \delta)k - k' - \Phi(k' - k) + d' - (1 + r^*)d - \frac{h^\omega}{\omega}) \\ + \beta E_{A'|A} v(d', k', A') \Big\}$$

- Business cycle moments very similar to EDEIR model (except autocorr. of *tb*).