International Reserve Management under Rollover Crises*

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Abstract

This paper investigates how a government should manage international reserves when it faces the risk of a rollover crisis. We ask, should the government accumulate reserves or reduce debt to make itself less vulnerable? We show that the optimal policy entails initially reducing debt, followed by a subsequent increase in both debt and reserves as the government approaches a safe zone. Furthermore, we find that issuing additional debt to accumulate reserves can lead to a reduction in sovereign spreads.

JEL classification: E4, E5, F32, F34, F41.

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1 Introduction

Governments are frequently exposed to episodes where investors suddenly lose confidence in the government's ability to repay the debt. This sudden loss of confidence can be self-fulfilling and result in situations where the government is unable to roll over the debt, and defaults on its obligations. Given the significant costs associated with rollover crises, understanding how governments can reduce their vulnerability is crucial for economic policy.

In this paper, we investigate whether governments should accumulate international reserves as a safeguard against rollover crises. A common argument is that reserves provide the government with liquid resources when it cannot roll over its debt. However, a government can also lower its vulnerability by reducing sovereign debt. To the extent that reserves earn a lower interest than the one paid on the debt, it is unclear whether building up a stock of reserves is an effective way to reduce the government's vulnerability to a rollover crisis.

We provide a simple environment to study the optimal management of international reserves for a government subject to the risk of rollover crises. We show that if the government is highly indebted, accumulating reserves is not optimal. Rather, the government should lower its vulnerability by reducing the debt. We show that once the debt is reduced significantly, accumulating reserves becomes strictly optimal. In particular, the government should finance the reserve accumulation by increasing the amount of borrowing. Notably, issuing more debt in this case *reduces* sovereign spreads.

Our environment is a canonical model of rollover crises, following Cole and Kehoe (2000), augmented with reserve accumulation. The government starts with an initial stock of reserves and long-duration debt, receives a constant stream of income, and discounts the future at the same rate as external investors. To abstract from the insurance motive highlighted in Bianchi, Hatchondo and Martinez (2018), we assume that the only source of uncertainty is the possibility of a rollover crisis. When the government is in good credit standing, it decides how much debt to issue, how many reserves to accumulate, and whether to repay the coupons due or default on its obligations. Upon default, the government faces a penalty and is permanently excluded from sovereign debt markets. However, it can keep its reserves, and continue to accumulate reserves in the future.

Our analysis begins by characterizing how the economy can be in one of three zones depending on the initial portfolio of debt and reserves. In the safe zone, the government repays the debt regardless of whether investors continue to roll over the debt, and thus it is not vulnerable to a run in equilibrium. In the default zone, the government finds it optimal to default regardless of whether investors are willing to lend. In the *crisis zone*, the government's default decision depends on investors' beliefs. If investors are willing to roll over the debt, it is optimal for the government to repay. On the other hand, if investors refuse to roll over the debt, the cost of repayment increases for the government, leading the government to default. In the crisis zone, the government is therefore vulnerable to self-fulfilling rollover crises, as in Cole and Kehoe (2000).

The key question we tackle is the following: Suppose a government is in the crisis zone. What is the best strategy to reach the safe zone? Should the government accumulate reserves or reduce its debt?

A key consideration for understanding our results is how gross positions affect the value of default and repayment for the government. An initial point to note is that higher reserves increase both the value of repayment and default. In contrast, higher debt reduces the value of repayment and does not affect the value of default, assuming full default. In an environment with one-period of debt, an increase in debt and reserves, both by one unit, leaves the value of repayment for the government unchanged. Because the value of default is increasing in reserves, an increase in debt and reserves makes the government more vulnerable to a rollover crisis by making default more attractive. In a scenario with one-period debt, accumulating reserves is therefore not optimal. The government should reduce the vulnerability to a rollover crisis by reducing debt.

When debt has long maturity, however, a simultaneous increase in debt and reserves helps the government relax its budget constraint in a situation where it is unable to borrow. This is because when debt has long maturity, only a fraction of the debt comes due every period. For a given net foreign asset position, an increase in debt and reserves therefore increases the resources available to the government and raises the value of repayment. As mentioned above, however, the value of default also increases with higher gross positions. Therefore, the overall effect of higher reserves and debt on the exposure to a rollover crisis is in principle ambiguous. Our analytical results elucidate when it is optimal to simply reduce the debt to reduce the vulnerability and when it is optimal to accumulate reserves.

Moreover, we show that the transition out of the crisis zone is *non-monotonic*. The optimal policy entails initially reducing debt, followed by a subsequent increase in both debt and reserves as the government exits the crisis zone. That is, when the government is highly indebted, it should raise the net foreign asset position exclusively through reductions in debt. The accumulation of reserves should be postponed until increasing reserves ensures that the government becomes safe from a rollover crisis. Interestingly, to exit the crisis zone, the government issues more debt to accumulate reserves, and this operation lowers sovereign spreads.

Related literature. Our paper belongs to the literature on international reserves. In particular, our paper is related to studies on the joint determination of reserves and defaultable sovereign debt that build on the workhorse model of sovereign default (Eaton and Gersovitz, 1981; Aguiar and Gopinath, 2006; and Arellano, 2008).[1]

Alfaro and Kanczuk (2009) study reserve accumulation in the canonical sovereign default model with one-period debt and show that it is not optimal for the government to accumulate reserves. This is because reserves make default more attractive and therefore worsen debt sustainability. In a model with long-term debt, Bianchi, Hatchondo and Martinez (2018) show that accumulating reserves provides insurance against negative income shocks that raise borrowing costs. In their model, when the government issues debt to accumulate reserves, it effectively reallocates resources from states with high bond prices and low marginal utility to states with low bond prices and high marginal utility. On the other hand, higher gross positions lead to higher sovereign spreads. The optimal portfolio for the government trades off the insurance benefits against the costs of higher spreads.[2] In contrast to these studies, we consider the possibility of rollover crises and show that issuing debt to accumulate reserves can reduce the vulnerability to rollover crises and lead to a decrease in sovereign spreads. Hernandez (2018) provides numerical simulations in a model with fundamental risk and rollover risk. Our approach focuses exclusively on rollover risk and allows us to provide an analytical characterization of when issuing debt to accumulate reserves is optimal.[3]

Corsetti and Maeng (2024) study the joint accumulation of reserves and debt in a model of belief-driven sovereign risk crises proposed by Aguiar, Chatterjee, Cole and Stangebye (2022). They show that accumulating reserves and one-period debt is desirable because it rules out an inefficient equilibrium with depressed bond prices.[4] The distinct role of reserves in our model is that they provide the government with liquidity in the event that investors refuse to roll over the debt in the future.

Aguiar and Amador (2014, 2024) and Bocola and Dovis (2019) study optimal debt maturity structure in a model with rollover crises. In these studies, a sufficiently long maturity can

[1]See Aguiar and Amador (2014) and Aguiar, Chatterjee, Cole and Stangebye (2016) for a review of the literature on sovereign debt and default.

[2]Bianchi and Sosa-Padilla (2023) show that under a fixed exchange rate regime, the hedging benefits lead to a macroeconomic stabilization motive for accumulating reserves.

[3]In his simulations, Hernandez (2018) finds negligible effects of changes in rollover risk on the accumulation of reserves, suggesting that the key motive for reserve accumulation in his model is related to fundamental risk instead of rollover risk. Our necessary condition for reserves to be optimal under rollover crisis does not appear to be satisfied in his calibration.

[4]Aguiar et al. (2022) show how the equilibrium with depressed bond prices can also be avoided by issuing low or high amounts of debt, relative to the good equilibrium level. Central to the mechanism in this framework is that intra-period uncertainty about the relative payoff between default and repayment induces a price schedule that is a non-monotonic function of debt.

prevent a rollover crisis, but it has the cost of exacerbating debt dilution.[5] Our results on the optimal deleveraging are connected in particular to those in Aguiar and Amador (2014), who show that it is optimal for the government to initially remain passive in long-term debt bonds—actively reducing short-term debt—and to lengthen the maturity at the end of the process. Aguiar and Amador (2024) study maturity swaps and show how they can improve bond prices and welfare under rollover risk. Our paper considers an exogenous maturity structure and studies the optimal accumulation of international reserves. One difference in our portfolio problem is that reserves increase the outside option of defaulting whereas the outside option of defaulting is exogenous in their model. The fact that reserves increase the value of default is important because this can potentially reduce the sustainable debt level, as in Bulow and Rogoff (1989).[6]

Conesa and Kehoe (2024) study a model of rollover crises where the government can commit to a tax rate in advance. They show that it is ex-ante optimal to set a high tax so that in the event of a rollover crisis, the government has sufficient tax revenue to repay the debt, thus deterring investors from running. However, this policy is suboptimal ex-post, as the government can borrow at the risk-free rate and would prefer to set a lower tax rate. They refer to this policy as "preemptive austerity."[7] In our model, reserves also serve a preemptive role, but the use of reserves to service the debt is ex-post optimal and does not require ex-ante commitment. Moreover, the accumulation of reserves allows the government to maintain a lower net foreign asset position and potentially higher consumption as it exits the crisis zone.

Outline. The remainder of the paper is organized as follows. Section 2 introduces the environment. Section 3 analyzes the optimal government portfolio. Section 4 concludes.

[5]The desirable properties of long-term debt are also highlighted in Cole and Kehoe (2000). See also Arellano and Ramanarayanan (2012) and Hatchondo, Martinez and Sosa-Padilla (2016) for studies of the maturity tradeoffs under fundamental risk. There is also an extensive literature on maturity tradeoffs in closed economies (Barro, 1999, 2003; Angeletos, 2002; Buera and Nicolini, 2004; Bhandari, Evans, Golosov and Sargent, 2017), where the key aspects involve distortionary taxation and fluctuations in the endogenous risk-free rate instead of default risk.

[6]Bulow and Rogoff (1989) show that lending cannot be supported in equilibrium in the absence of direct punishments when the government has the ability to save abroad in reserves.

[7]See also a related literature on sovereign debt crises that followed the Eurozone crisis (e.g., Aguiar, Amador, Farhi and Gopinath, 2013, 2015; Bacchetta, Perazzi and Van Wincoop, 2018; Corsetti and Dedola, 2016; Corsetti and Maeng, 2023; Araujo, Leon and Santos, 2013; Lorenzoni and Werning, 2019; Ayres, Navarro, Nicolini and Teles, 2018, 2023; Camous and Cooper, 2019; Bassetto and Galli, 2019; Cole, Neuhann and Ordonez, 2024; Bianchi and Mondragon, 2022; Bianchi, Ottonello and Presno, 2023).

2 Environment

Consider a small open economy with time indexed by $t = 0, 1....$ There is a single consumption good, which is freely tradable. The government in the small open economy receives a constant endowment y of the consumption good and trades bonds, b, and risk-free assets, a, with a continuum of external investors. Investors are risk-neutral and share the same discount factor, $\beta \in (0, 1)$, as the government.

2.1 Government Problem

The government's preferences are represented by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t [u(c_t) - \phi d_t],$$

where c_t denotes consumption and $u(\cdot)$ is an increasing, twice-continuously differentiable, and strictly concave function. We denote by d_t the default decision that takes the value of 1 if the government is in default and 0 otherwise.

The government enters every period with an initial portfolio of assets and outstanding bonds (a, b). We refer to the assets as "reserves" and assume without loss of generality that they have a one-period maturity.[8] The return on reserves is $1 + r = \beta^{-1}$.

The bonds the government issues are long-maturity bonds with geometrically decaying coupons, as in Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012). A bond issued at any date t promises to pay $\left(\frac{\delta+r}{1+r}\right)[1, (1-\delta), (1-\delta)^2, ...]$ in periods $t+1, t+2, t+3....$ Notice that the coupon payments are normalized so that the price of the bond equals $1/(1+r)$ if there is no default risk. The Macaulay duration of the bond is parameterized by $1/\delta$.

If the government has not defaulted in the past, we can write its budget constraint when it chooses to repay in period t as:

$$c_t = y + a_t - \left(\frac{\delta + r}{1 + r}\right) b_t - \frac{a_{t+1}}{1 + r} + q_t[b_{t+1} - (1 - \delta)b_t].$$

That is, the government collects its income and assets, consumes, pays the debt coupons, accumulates reserves, and issues bonds at a price schedule q_t. As we will see, in the Markov

[8] When reserves can be sold costlessly in a spot market and there are no fluctuations in the risk-free interest rate, it is equivalent whether reserves are one-period or long-term assets. The equivalence would break if there were shocks to the risk-free rate.

equilibrium, the bond price schedule will depend on the portfolio chosen by the government.

If the government defaults on its debt, it is excluded permanently from financial markets and faces a utility cost ϕ every period.[9] As occurs in practice, the government is able to keep its holdings of reserves and continue to adjust them over time.[10] The budget constraint in case of default is as follows:

$$c_t = y + a_t - \frac{a_{t+1}}{1+r}.$$

Timing. The timing follows Cole and Kehoe (2000). We assume the government makes a repayment decision after borrowing and reserve accumulation have taken place. This differs from the Eaton-Gersovitz timing, in which the government first chooses to repay or default and then chooses its portfolio (see Aguiar and Amador, 2014). The fact that investors are atomistic will open the door to coordination failures where a *good equilibrium* in which investors continue to roll over the bonds and the government repays coexists with a *bad equilibrium* where investors refuse to roll over the bonds and the government defaults.

We will use ζ to denote a sunspot variable that will determine the type of equilibrium. If $\zeta = 0$, investors will expect others to continue rolling over the bonds, while if $\zeta = 1$, investors will expect others to stop rolling over the bonds. The probability that $\zeta = 1$ is constant and denoted by λ. As we will see, whether a rollover crisis actually takes place will be determined endogenously and will be a function of the initial portfolio of the government.

Recursive problem. If the government has not defaulted in the past, it chooses whether to repay or default, and its value function is given by

$$V(a, b, \zeta) = \max\{V_R(a, b, \zeta), V_D(a)\}, \tag{1}$$

where V_D and V_R represent, respectively, the values of default and repayment. We assume without loss of generality that if the government is indifferent between repaying and defaulting, it repays.

[9]Our results about the optimal portfolio are qualitatively the same if we impose an income cost of defaulting.

[10]Under the Foreign Sovereign Immunities Act (FSIA), reserves cannot be legally seized by creditors.

The value of the government under default is given by

$$V_D(a) = \max_{c,a' \geq 0} \left\{ u(c) - \phi + \beta V_D(a') \right\}, \tag{2}$$

$$\text{subject to}$$

$$c = y + a - \frac{a'}{1+r}.$$

The value of repayment is given by

$$V_R(a,b,\zeta) = \max_{c,a' \geq 0, b'} \left\{ u(c) + \beta \, \mathbb{E} V(a',b',\zeta') \right\}, \tag{3}$$

$$\text{subject to}$$

$$c = y + a - \left(\frac{\delta + r}{1+r} \right) b - \frac{a'}{1+r} + q(a',b',s) \left(b' - (1-\delta) b \right),$$

where we index the price schedule q by the initial state $s = (a, b, \zeta)$ to capture the possibility of multiple equilibria, as will become clear below. The expectation operator $\mathbb{E}$ is taken over the realization of the next-period sunspot ζ'.

2.2 Markov Equilibrium

Before defining the equilibrium, we present the no-arbitrage condition imposed by the presence of risk-neutral international investors. Investors' optimality implies that they equate the return from a risk-free asset to the expected return from government bonds. That is,

$$q(a',b',s) = \begin{cases} \frac{1}{1+r} \mathbb{E} \left[(1-d') \left(\frac{\delta+r}{1+r} + (1-\delta) q(a'',b'',s') \right) \right] & \text{if } d(s) = 0, \\ 0 & \text{if } d(s) = 1, \end{cases} \tag{4}$$

where $b'' = \hat{b}(a',b',\zeta')$, $a'' = \hat{a}(a',b',\zeta')$, and $d' = \hat{d}(a',b',\zeta')$ represent the policies the government is expected to follow in the next period. If the government is expected to default at the end of the current period, the bond price is zero. If the government is expected to repay in the current period, the bond price takes into account the coupon payments tomorrow and the secondary market price of the bond (which depends on future probabilities of default).

The Markov equilibrium in this economy is defined as follows:

Definition 1. A Markov equilibrium is defined by a set of policies for the government $\{c(\cdot), d(\cdot), a'(\cdot), b'(\cdot)\}$ and a bond price schedule $q(\cdot)$ such that

i) the policies solve the government problem given the bond price schedule;

ii) the price schedule satisfies (4) given the government policies.

2.3 Multiplicity of Equilibria

As in Cole and Kehoe (2000), the government can be subject to rollover crises where the government defaults on its debt because investors stop rolling over the bonds. To determine the states in which the government is vulnerable to a rollover crisis, we need to distinguish between two scenarios, one where investors continue rolling over the bonds and one where they do not.

Consider first a situation in which each individual investor expects others to continue rolling over the bonds. In this case, the government solves the following problem:

$$V_R^+(a,b) = \max_{c,a'\geq 0,b'} \left\{u(c) + \beta\,\mathbb{E}V(a',b',\zeta')\right\}, \tag{5}$$
$$\text{subject to}$$
$$c = y + a - \left(\frac{\delta+r}{1+r}\right)b - \frac{a'}{1+r} + \tilde{q}(a',b')\left(b' - (1-\delta)b\right),$$

where $\tilde{q}$ denotes the "fundamental" bond price:

$$\tilde{q}(a',b') = \frac{1}{1+r}\mathbb{E}\left\{(1-d')\left[\left(\frac{\delta+r}{1+r}\right) + (1-\delta)\,q(a'',b'',s')\right]\right\}. \tag{6}$$

Consider now a situation in which the government would like to issue new debt, but investors are unwilling to lend. In this case, the value of repayment for the government is given by

$$V_R^-(a,b) = \max_{c,a'\geq 0} \left\{u(c) + \beta\,\mathbb{E}V(a',(1-\delta)b,\zeta')\right\}, \tag{7}$$
$$\text{subject to}$$
$$c = y + a - \left(\frac{\delta+r}{1+r}\right)b - \frac{a'}{1+r}.$$

An immediate implication is that $V_R^+(a,b) \geq V_R^-(a,b)$. That is, a government that can roll over the debt obtains at least the same value as a government that cannot.[11]

When $V_R^-(a,b) < V_D(a) \leq V_R^+(a,b)$, we have multiple equilibria. If the government faces the fundamental price $\tilde{q}$, then the government repays. If the government faces a price $q = 0$, then the government defaults. We examine below for which initial portfolios self-fulfilling debt crises are possible.

The safe zone, the crisis zone, and the default zone. Given the value functions (2), (5), and (7), we can split the economy in three different zones depending on the initial portfolio (a,b):

$$\begin{aligned}
\mathbf{S} &= \left\{(a,b) : V_D(a) \leq V_R^-(a,b)\right\}, \\
\mathbf{D} &= \left\{(a,b) : V_D(a) > V_R^+(a,b)\right\}, \\
\mathbf{C} &= \left\{(a,b) : V_R^-(a,b) < V_D(a) \leq V_R^+(a,b)\right\}.
\end{aligned}$$

$\mathbf{S}$ is the safe zone: the government is better off repaying, regardless of the sunspot realization. $\mathbf{D}$ is the default zone: the government is better off defaulting, regardless of the sunspot realization. $\mathbf{C}$ is the crisis zone: the government finds it optimal to repay if lenders are willing to lend, while it finds it optimal to default if the lenders are not willing to lend.

Debt thresholds. The value functions of repayment are strictly decreasing in debt and strictly increasing in reserves and continuous in both. We can then define thresholds $b^-(a)$ and $b^+(a)$ that determine respectively the boundary between $\mathbf{S}$ and $\mathbf{C}$ and the boundary between the $\mathbf{C}$ and $\mathbf{D}$, for every a. That is, $b^-(a)$ is given by

$$V_R^-(a, b^-(a)) = V_D(a), \tag{8}$$

and $b^+(a)$ is given by

$$V_R^+(a, b^+(a)) = V_D(a). \tag{9}$$

Notice that these two thresholds are uniquely determined for any given a, and so we have constructed two functions that map an initial level of reserves to a level of debt that makes the government indifferent between repaying and defaulting depending on whether investors are willing to roll over the debt or not. A central aspect of our analysis will be how these thresholds vary with reserves and what this implies for the optimal portfolio.

[11] One element implicit in the budget constraint in problem (7) is that if the government were to repurchase debt when investors are unwilling to lend, the price of bonds would rise to the fundamental price (see Aguiar and Amador, 2014).

2.4 Equilibrium Payoffs

Given the definition of $\mathbf{S}, \mathbf{D}$, and $\mathbf{C}$, we can write the payoff for the government as

$$V(a,b,\zeta) = \begin{cases} V_R^+(a,b) & \text{if } (a,b) \in \mathbf{S}, \\ V_R^+(a,b) & \text{if } (a,b) \in \mathbf{C} \ \& \ \zeta \in \{0\}, \\ V_D(a) & \text{if } (a,b) \in \mathbf{C} \ \& \ \zeta \in \{1\}, \\ V_D(a) & \text{if } (a,b) \in \mathbf{D}. \end{cases} \tag{10}$$

When $(a,b) \in \mathbf{C}$, the equilibrium outcome is undetermined and the government's payoff depends on the sunspot. Notice that the equilibrium payoff for the government never takes the value V_R^-, as this is an off-equilibrium payoff. Nonetheless, V_R^- is essential to determine which zone the government is in and thus, the ultimate payoff for the government.

We now analyze the values in each zone.

Safe zone. In Cole and Kehoe (2000) under $\beta(1+r) = 1$, when the government is in the safe zone, it stays in the safe zone with a constant level of debt and consumption. The logic is that once the government reaches the safe zone, it can achieve the level of consumption that would prevail in the absence of default risk. Crucial for this result is that if the government stays in the safe zone, it can borrow at the risk-free rate. If the government were to choose a portfolio outside the safe zone, the bond price would fall, reflecting the positive probability of default, and the government would incur the expected costs of defaulting.

In our model with reserves, maintaining a constant level of consumption implies that $c = y + (1-\beta)(a-b)$ and any portfolio $a' - b' = a - b$ such that $(a,b) \in \mathbf{S}$ guarantees this to be the case. While it is immediate that any other portfolio in the safe zone that deviates from a stationary consumption delivers strictly lower utility (because of concavity), we highlight that when the government can save in reserves, it is not a general result that the government necessarily stays in the safe zone. For a high enough δ, however, it is possible to establish that the government does find it optimal to stay in the safe zone.[12] We proceed below under

[12] A sufficient condition for the optimality of "staying put" is that $b(1-\delta) \leq b^+\big(a - \frac{\delta+r}{1+r}b\big)$. In results available upon request, we show that in the case of a perpetuity (i.e., $\delta = 0$) if the government is indifferent between staying put and defaulting today, it is strictly optimal for the government to repay the debt today and *default tomorrow* with probability one. The intuition for this perhaps surprising result is that reserves allow the government to postpone the default costs and improve consumption smoothing. In this regard, the result is connected with Bulow and Rogoff (1989) who show how access to reserves reduces debt sustainability.

this operating assumption and thus postulate that the value in the safe zone is given by

$$V(a,b) = \frac{u(y + (1-\beta)(a-b))}{1-\beta} \equiv V_S(a-b), \tag{11}$$

and the policy functions satisfy

$$\begin{aligned} c &= y + (1-\beta)(a-b), \\ a' - b' &= a - b \end{aligned}$$

with $(a', b') \in \mathbf{S}$. Notice that the payoff for the government depends only on the net foreign asset position (NFA), defined as $a - b$, and not on the gross positions. Moreover, while the choice for consumption and the NFA are unique in the safe zone, there are generically multiple portfolios that are optimal. That is, as long as the new portfolio keeps the same NFA and the government remains in the safe zone next period, the portfolio is (weakly) optimal. The value in the safe zone, which we denote by V_S, is therefore only a function of the NFA.

Default zone. Given our assumption that $\beta(1+r) = 1$, under default, the government keeps reserves constant and consumes its income plus the annuity value of the reserves. Thus, the value of default is given by

$$V_D(a) = \frac{u(y + (1-\beta)a) - \phi}{1-\beta}. \tag{12}$$

Crisis zone. Recall that the problem the government faces when investors are willing to roll over the bonds is given by (5). Using the results above, we can write the continuation value for the government as follows:

$$\mathbb{E}V(a', b', \zeta') = \begin{cases} V_S(a' - b') & \text{if } (a', b') \in \mathbf{S}, \\ (1-\lambda)V_R^+(a', b') + \lambda V_D(a') & \text{if } (a', b') \in \mathbf{C}, \\ V_D(a') & \text{if } (a', b') \in \mathbf{D}. \end{cases} \tag{13}$$

That is, if the government chooses a portfolio in the safe zone, the continuation value is given by $V_S(a' - b')$. If the government chooses a portfolio in the crisis zone, then the government will obtain $V_D(a')$ with probability λ and $V_R^+(a', b')$ with probability $1 - \lambda$. Finally, if the government chooses a portfolio in the default zone, its continuation value is given by $V_D(a')$.

Staying in the crisis zone is costly for the government. If the government faces a good sunspot tomorrow, it pays ex post a high interest rate on the debt. If the government faces a bad sunspot tomorrow, the government does not pay the debt but it faces the cost of defaulting. While the government pays an actuarily fair interest rate to investors, it still bears the cost of defaulting. As shown by Cole and Kehoe (2000), the government has therefore incentives to deleverage until it reaches the safe zone. Exiting the crisis zone, however, is also costly. To the extent that the utility function is strictly concave, the government may try to exit the crisis zone slowly. Depending on the sunspot realization, the government may be able to reach the safe zone eventually, or it may default along the way. How fast the government attempts to exit depends on the perceived probability of facing the bad sunspot. In particular, a higher λ induces a faster exit of the crisis zone.

Using that the safe zone is an absorbent state, we can iterate on investors' break-even condition (6) to arrive at an expression for the bond price faced by the government. In particular, we can use that the bond price becomes $q = 1/(1+r)$ the period in which the government chooses a portfolio that takes it to the safe zone in the next period. Starting from $t = 0$, suppose that the portfolio choice today is (a', b') and the policy function of the government is such that the government reaches the safe zone in period T so long as $\{\zeta_t\}_{t=0}^{T-1}$. We have that for any $T > 0$ the bond price is given by

$$q(a', b') = \frac{\delta + r}{1+r} \sum_{t=1}^{T-1} \left(\frac{1-\lambda}{1+r} \right)^t (1-\delta)^{t-1} + \left[\frac{(1-\lambda)(1-\delta)}{1+r} \right]^{T-1} \frac{1}{1+r} \tag{14}$$

The first term captures the bond coupon payments investors expect to receive, and the second term reflects the risk-free price of the bond once the government exits the crisis zone.

In the Cole and Kehoe model with one-period debt, consumption is constant over time while the government tries to exit the crisis zone, and once the government reaches the safe zone, consumption increases and then stays constant thereafter. The idea is that keeping consumption low allows the government to reduce its debt and reach the safe zone—and once the government is safe, debt can be kept constant. Proposition 1 provides a similar result in our economy.

Proposition 1 (Monotonically increasing consumption path)**.** *Consider an initial portfolio* $(a_0, b_0) \in \mathbf{C}$ *such that the government exit time is* T*. Then, if* $\zeta_t = 0$ *for all* $t \leq T-1$*, we have* $c_{t+1} \geq c_t$ *for all* $t \leq T$*.*

Proof. In Appendix A.1. □

A subtle difference with the canonical model is that consumption may be strictly increasing as the government tries to exit the crisis zone. This result is due to the dilution effect from long-term debt. In particular, the bond price at which investors are willing to lend depends on the expected T at which the government will exit (see eq. 14). Therefore, if keeping consumption low today relative to tomorrow reduces the exit time T, the government can obtain a better price on the debt. As we will see in the numerical simulation, consumption will be constant between two periods for which a small change in the portfolio would not affect the exit time while and increasing otherwise.[13]

2.5 Solving for the debt thresholds

In general, solving for the two debt thresholds, $b^-(a)$ and $b^+(a)$, requires solving for a fixed point. To solve for V_R^+ (and thus $b^+(a)$), we need to determine the default probability as a function of the portfolio, which depends on the two thresholds. To solve for V_R^- (and thus $b^-(a)$), we need to know in general V_R^+ as this represents the continuation value after repaying today. However, if a government that is currently facing a run and repays the coupons due today is no longer vulnerable in the future, then $b^-(a)$ can be computed directly from the following condition:

$$\max_{a'\geq 0} u\left(y - \left(\frac{\delta + r}{1+r}\right) b^-(a) + a - \frac{a'}{1+r}\right) + \beta V_S(a' - (1-\delta)b^-(a)) = V_D(a), \qquad (15)$$

The continuation value on the left-hand side reflects the assumption that the government is safe going forward. Notice that the validity of this assumption is guaranteed for δ close to one. Intuitively, if δ is close to one, there is little debt left to make a government vulnerable after repaying the coupons due.[14]

We have the following results:

Lemma 1. *Consider an initial portfolio (a, b). We then have that*

(i) the solution to the left-hand side problem in (15) is $a'(a, b) = \max[0, a - \delta b]$;

(ii) if $a \geq \delta b$ and $(a - \delta b, (1-\delta)b) \in \mathbf{S}$, then $V_R^-(a, b) = V_R^+(a, b)$.

Proof. In Appendix A.2 □

[13] In a continuous time model with outside option shocks and long-term bonds, Aguiar and Amador (2023) find that consumption is *strictly* increasing over time as the government tries to reach the safe zone. This is because the exit time in their model is continuous and strictly increasing in debt within the crisis zone.

[14] Notice also that if the government were to start from $b^-(0)$, the government would be safe tomorrow with $a' = 0$. To see that $(0, (1-\delta)b^-(0)) \in \mathbf{S}$, notice that if a government is indifferent between repaying and defaulting with the portfolio $(0, b^-(0))$, it will find it strictly optimal to repay tomorrow given a portfolio with a lower amount of debt, (i.e., $(0, (1-\delta)b^-(0))$).

Result (i) says that when the government faces a run today and is safe tomorrow, it will be at a corner with zero reserves if $a < \delta b$. The intuition is that when the government cannot roll over the bonds and starts with low reserves, it chooses the minimum reserves possible if it anticipates that it will be safe in the future. Notice that the solution to problem (7) is not in general the same as the one for (15). In particular, if the government remains vulnerable tomorrow after repaying during a run, it has more incentives to keep positive reserves.

Result (ii) says that if the government has a high level of reserves, it can achieve the unconstrained level of consumption even when investors refuse to roll over the debt. Notice that the result requires that the government is not vulnerable tomorrow after repaying the coupons due today. That is, it is possible that a government has enough reserves to pay the coupons due, achieve the same level of consumption as when the government can rollover the debt, keep positive reserves, and still have $V_R^-(a,b) < V_R^+(a,b)$. This is because the fall in reserves necessary to make the coupon payments today may make the government vulnerable tomorrow. In turn, this implies a reduction in the value from repaying today in the event of a run.

One interesting observation from (ii) is that reserves do not need to be large enough to pay for all coupons in order to be safe from a rollover crisis. That is, given $r > 0$, there exists $a \in [\delta b, \left(\frac{\delta+r}{1+r}\right) b]$ such that the government is safe. To be safe, the government needs just enough reserves to repay the fraction of the debt that would allow the government to keep the NFA constant. In effect, when the government has a negative NFA, it uses a fraction of the endowment to pay the interest on the debt.[15]

3 Reserve Management under Rollover Crises

In this section, we turn to the main question we investigate in this paper: *How should the government choose its portfolio to manage the risk of a rollover crisis?*

3.1 The Three Zones

We start the analysis by showing how the initial portfolio determines which zone the economy is in. Figure 1 presents the results.[16] The blue-solid line and the red-dashed line denote

[15]To see this more clearly, suppose the government starts with the portfolio $(\delta b, b)$ and thus NFA $-(1-\delta)b$. If the government cannot roll over the debt, $b' = (1-\delta)b$. By setting $a' = 0$, the government keeps the same NFA and achieves the ideal level of consumption, which in this case is $c = y - (1-\beta)(1-\delta)b$.

[16]Throughout, we use calibrated parameter values listed in Table 1, to be described below.

respectively $b^+(a)$ and $b^-(a)$. The two lines indicate the boundaries between the crisis zone and the default zone and between the crisis zone and the safe zone. Recall that a government that starts with a portfolio $(a, b^-(a))$ is indifferent between defaulting and repaying when it cannot roll over the bonds, while a government that starts with a portfolio $(a, b^+(a))$ is indifferent between defaulting and repaying when it can roll over the bonds. Thus, we must have $b^+(a) \geq b^-(a)$, as observed in the figure. We can also see that $b^+(a) = b^-(a)$ for large values of a, a result in line with part (ii) of Lemma 1. That is, for large a, the crisis zone is empty. Defaults only happen due to fundamentals.

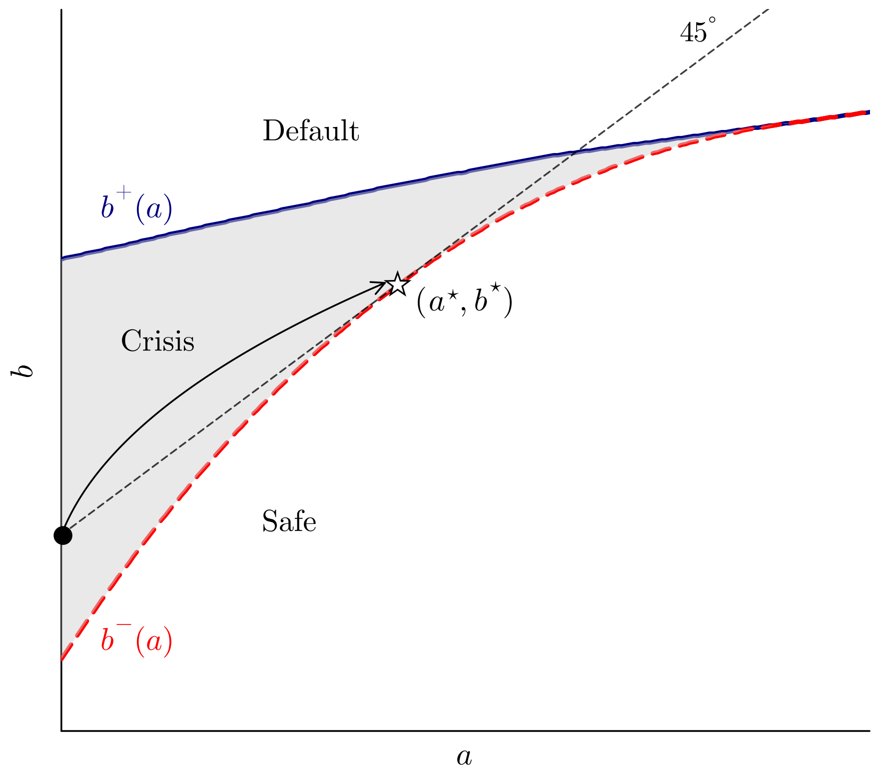

Figure 1: The three zones. The boundaries $b^-(a)$ and $b^+(a)$ are defined in (8) and in (9), respectively.

Consider any pair (a, b) at the $b^-(a)$ boundary. Holding a fixed, we can see that a lower level of debt takes the government to the safe zone, and a higher level of debt takes it to the crisis zone. Moreover, we can see that $b^-(a)$ has a positive slope: holding b fixed, a higher level of reserves takes the government to the safe zone, while a lower level of reserves puts the government in the crisis zone. The figure shows that $b^+(a)$ also has a positive slope, but it is less steep than $b^-(a)$. To understand better this results, we differentiate (8) and (9) with respect to reserves:

$$\frac{\partial b^{-}(a)}{\partial a} = \frac{\frac{\partial V_D(a)}{\partial a} - \frac{\partial V_R^{-}\left(a,b^{-}(a)\right)}{\partial a}}{\frac{\partial V_R^{-}(a,b^{-}(a))}{\partial b}}, \qquad \frac{\partial b^{+}(a)}{\partial a} = \frac{\frac{\partial V_D(a)}{\partial a} - \frac{\partial V_R^{+}\left(a,b^{+}(a)\right)}{\partial a}}{\frac{\partial V_R^{+}(a,b^{+}(a))}{\partial b}}. \tag{16}$$

That is, the slopes of b^- and b^+ are determined by how much the value functions of repayment and default change when we vary a and b at the indifference points.

As we show formally below, both numerators in (16) are positive. For the case of b^-, the result follows because when debt cannot be rolled over, making the debt payment forces an increase in the government's marginal utility of consumption. For the case of b^+, the result follows because when the government can borrow, it reduces consumption to try to exit the crisis zone, as we showed in Proposition 1. Given that the denominator in (16) is negative, this implies that both slopes are positive. The reason why the slope of $b^-(a)$ is steeper than $b^+(a)$ is that when reserves increase, the value of repayment goes up more when the government cannot roll over the debt. In effect, the government becomes borrowing-constrained, which leads to a higher marginal utility.

We summarize these results in the following proposition.

Proposition 2 (Monotonicity)**.** *In any Markov equilibrium, $b^-(\cdot)$ and $b^+(\cdot)$ are increasing in a, for all a. In addition, $\frac{\partial b^-(a)}{\partial a} \geq \frac{\partial b^+(a)}{\partial a}$ with strict inequality for a such that $b^-(a) < b^+(a)$.*

Proof. In Appendix A.3 □

We have seen that reserves help expand the safe zone. Even though this suggests that reserves could be desirable, the government in principle, could also reduce its vulnerability by decreasing debt instead. The key issue is whether increasing both debt and reserves leads to an increase in $V_R^-(a, b)$ that offsets the increase in $V_D(a)$. More concretely, consider a government portfolio that lies slightly above the red-dashed curve in Figure 1. Does an increase in debt and reserves by one unit push the government into the crisis zone or the safe zone? We examine this question next.

3.2 A Joint Increase in Debt and Reserves

Each of the portfolios in the $b^-(a)$ boundary is associated with a different level of net foreign assets. It will be useful to examine the portfolio in the safe zone with the lowest NFA position. We refer to this portfolio as the "lowest-NFA safe portfolio", and we denote it by $(a^\star, b^\star)$.

Definition 2. The lowest-NFA safe portfolio $(a^\star, b^\star)$ is the portfolio in the safe zone with the lowest net foreign asset position.

Formally, we have that the lowest-NFA safe portfolio is given by

$$(a^\star, b^\star) = \underset{a \geq 0, b}{\operatorname{argmin}} \; a - b \qquad \text{s.t.} \quad (a, b) \in \mathbf{S}. \tag{17}$$

Using that $(a, b) \in \mathbf{S}$ if $b \leq b^-(a)$ and assuming a *strictly interior solution for* $a^\star$, we obtain that the lowest-NFA safe portfolio satisfies $\frac{\partial b^-(a^\star)}{\partial a} = 1$. Graphically, this portfolio corresponds to the point highlighted in Figure 1 where the 45° line is tangent to the red-dashed line. It is then immediate that if the government starts from a point on or below this tangent line, it can move to the safe zone by increasing reserves and debt by the same amount (thus keeping the same NFA). The arrow in the figure illustrates how a government at the tangent line with zero initial reserves can jump to the safe zone by choosing $(a^\star, b^\star)$.

The lowest-NFA safe portfolio will constitute a focal point. When a government is deep in the crisis zone (i.e., it is above the aforementioned tangent line), it needs to increase its NFA to reach the safe zone. The larger the required increase in the NFA, the higher the cost of exiting the crisis zone because of the concavity of the utility function. The portfolio $(a^\star, b^\star)$ makes the government safe and minimizes the need for consumption cuts.

The key question we tackle next is when do we have $a^\star > 0$?

A condition for $a^\star > 0$. The proposition below provides a sufficient condition guaranteeing that the solution to (17) is strictly interior.

Proposition 3 (Positive reserves)**.** *Suppose that the boundary of the crisis region at zero reserves $b^-(0)$ satisfies*

$$\beta(1-\delta)\left[u'\left(y - \left(\frac{\delta + r}{1+r}\right) b^-(0)\right) - u'\left(y - (1-\beta)(1-\delta) b^-(0)\right)\right] > u'(y). \tag{18}$$

Then, the lowest-NFA safe portfolio features a strictly positive level of reserves. That is, $a^\star > 0$.

Proof. In Appendix A.4 □

The argument is that when (18) holds, the slope of $b^-(a)$ exceeds 1 at zero reserves. That is, starting from a point with zero reserves where the government is indifferent between repaying and defaulting, an increase in reserves of one unit implies that there exists an increase in debt of more than one unit that keeps the government indifferent.

As condition (18) indicates, a key determinant for the lowest-NFA safe portfolio featuring strictly positive reserves is the curvature of the utility function. Intuitively, when the government cannot roll over the bonds, this implies a reduction in consumption. When the utility function is highly concave, this means that there is a large drop in utility, and thus a large drop in V_R^-. In this situation, a higher level of reserves has a particularly high marginal utility and can offset the fact that higher reserves also increase the value of default.

Maturity also plays a critical role for condition (18) to hold. In particular, the condition can only be satisfied for intermediate values of maturity. In the case of $\delta = 0$,, the bond becomes a perpetuity, in which case rollover risk becomes irrelevant as the government only pays interest and never pays any principal. Conversely, in the case of $\delta = 1$, condition (18) cannot be satisfied. Intuitively, when debt is one-period, the value of repayment for the government depends only on $a - b$ (i.e., it is independent of the gross positions). Because V_D increases with a, it thus follows that a one-unit increase in a and b must lower $V_R^-(a, b) - V_D(a)$. This implies that if a government is indifferent between repaying while facing a run and defaulting, an increase in debt and reserves will always push the economy into the crisis zone. The proposition below formalizes that if $\delta = 1$, the lowest NFA-safe portfolio must feature $a^\star = 0$.

Proposition 4. *Suppose that $\delta = 0$ or $\delta = 1$. Then, the lowest-NFA safe portfolio features $a^\star = 0$.*

Proof. In Appendix A.5. □

3.3 Exiting the Crisis Zone

We analyze now what the best strategy is for a government that is trying to exit the crisis zone: Should the government reduce its debt or increase reserves? If reserves are optimal, should the government slowly build up its stock of reserves?

Iso-T regions. We first examine how long it takes for the government to exit the crisis zone depending on the initial portfolio. Following the terminology of Aguiar and Amador (2013), we construct "Iso-T regions." Starting from $t = 0$, we compute in which period the government will reach the safe zone (so long as it does not get hit by a bad sunspot before then).

Figure 2 shows the exit time T as a function of the initial portfolios (a, b). As we can see, when the gross positions are close to $(a^\star, b^\star)$, the government exists in one period. That is, $T = 1$. As we move up and toward the left (that is, increasing debt and lowering reserves), it takes more periods to exit. For this parameterization, we can see a region with $T = 2, T = 3$, and $T = 4$. When the government starts from debt levels above $b^+(a)$, the government strictly prefers to default in equilibrium.

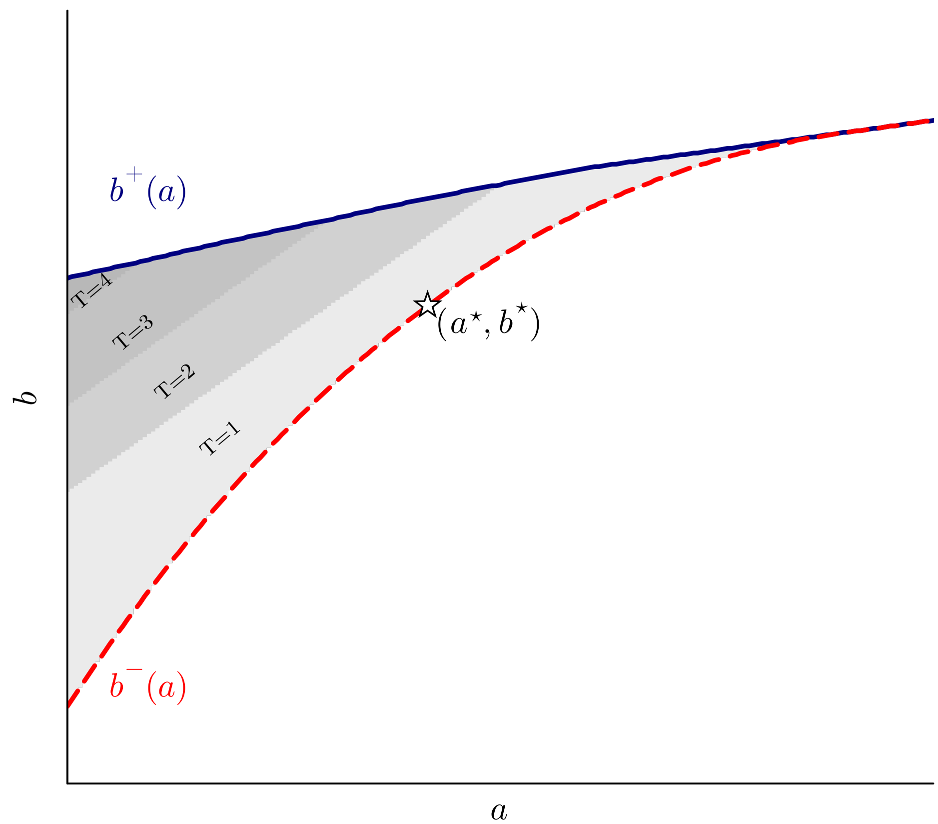

Figure 2: Iso-T Regions. The shaded areas describe how many periods it takes for the government to exit the crisis zone.

Optimal portfolio. We now analyze the optimal portfolio choice for the government that is today in the crisis zone and is able to borrow. The proposition below characterizes the solution.

Proposition 5 (Optimal portfolio)**.** *Consider an initial* $(a, b) \in \mathbf{C}$*. The optimal portfolio choice* $\{a'(a, b), b'(a, b)\}$ *that solves* (5) *satisfies the following conditions:*

(i) Suppose that $a - b < a^\star - b^\star$. Then, if $(a'(a,b), b'(a,b)) \in \mathbf{S}$ we have that $a'(a,b) = a^\star$ and $b'(a,b) = b^\star$.

(ii) Suppose that $(a'(a,b), b'(a,b)) \in \mathbf{C}$. Then, we have that $a'(a,b) = 0$.

Proof. In Appendix A.6. □

Part (i) considers a government in the crisis zone that has an NFA that is lower than the lowest-NFA safe portfolio. The proposition shows that if the government chooses a portfolio that puts it in the safe zone, then the government chooses the lowest-NFA safe portfolio $(a^\star, b^\star)$. The idea is that choosing $(a^\star, b^\star)$ allows the government to exit the crisis zone minimizing the cut in consumption. Portfolios in the safe zone with lower amounts of reserves would imply the government needs to reduce debt by more than the reduction in reserves. Portfolios in the safe zone with higher amounts of reserves would allow the government to borrow more but the increase in borrowing would be smaller than the increase in reserves.

Part (ii) considers a situation where the government chooses a portfolio that keeps it in the crisis zone in the next period. The proposition shows that it is not optimal to accumulate reserves in this case. To understand the intuition for this result consider an alternative strategy where the government issues more debt and accumulates reserves while keeping consumption constant. Notice that because the bond price is lower than $\frac{1}{1+r}$, the government's face value of debt increases by more than the reserves accumulated. Assume further that the number of periods it takes to exit does not change, thus implying the same bond price. Following this strategy implies that the government would have more resources available in the event of default but fewer resources available in the event of repayment. Crucially, because the government is raising its NFA while it attempts to exit the crisis zone, this implies that the marginal utility is higher under repayment than under default.[17] Therefore, issuing debt to accumulate reserves in the crisis zone is strictly suboptimal as long as the government remains in the crisis zone.

Putting these two parts together establishes that *the optimal exit strategy is to delay the accumulation of reserves until the government is ready to exit the crisis zone.* We summarize this in the following corollary.

Corollary 1 (Optimal exit strategy)**.** *Consider a portfolio $(a,b) \in \mathbf{C}$ such that $a - b < a^\star - b^\star$ and the government exits after T periods for $T < \infty$, provided that $\{\zeta_t\}_{t=0}^{T-1} = 0$. Then, we have $a_{t+1} = 0$ for all $t < T-1$ and $a_T = a^\star$, $b_T = b^\star$.*

[17] A reader may wonder whether this result relies on the assumption that the cost of default is in terms of utility and not resources. We can show, however, that the same result holds under an income cost of defaulting.

Proof. In Appendix A.7. □

We turn next to the case when the government has an initial NFA higher than the one in the lowest-NFA safe portfolio. The proposition below shows that the government exits the crisis zone in one period.

Proposition 6 (Immediate exit)**.** *Suppose that condition* (18) *holds. Consider an initial portfolio* $(a, b) \in \mathbf{C}$ *and suppose that* $a - b > a^\star - b^\star$*. Then, the government exists in one period. Moreover, any portfolio* (a', b') *such that* $(a', b') \in \mathbf{S}$ *and* $a - b = a' - b'$ *is optimal.*

Proof. In Appendix A.8. □

The logic for why the government exits in one period is that it is feasible for the government to choose a portfolio that takes it to the safe zone without having to reduce consumption relative to the unconstrained optimal. That is, exiting the crisis zone is not costly in this situation. Notice that in this case, there are a range of portfolios that are optimal, including in particular, $(a^\star, b + a^\star - a)$.

Illustration. In Figure 3 we show three simulations, one where the government exits in one period (panel [a]), one where it exits in two periods (panel [b]), and one where it exits in three periods (panel [c]). As is consistent with Proposition 5, we can see that when the government exits in two periods, it chooses zero reserves in the first period, and when the government exits in three periods, it chooses zero reserves in periods 1 and 2.

Panel (d) of Figure 3 shows all the possible portfolios chosen throughout the transition for the entire range of possible initial portfolios with an NFA lower than the lowest-NFA safe portfolio. As the figure illustrates, the government never accumulates reserves unless doing so takes it to the safe zone. It is also interesting to note that there are "holes" in the figure in the sense that there are a range of b' values that are never chosen by the government in equilibrium. This is because, for those levels, the government is better off either borrowing slightly less and exiting in fewer periods (thus, obtaining a higher bond price) or borrowing slightly more and exiting in the same number of periods (thus, achieving a higher intertemporal smoothing of consumption).

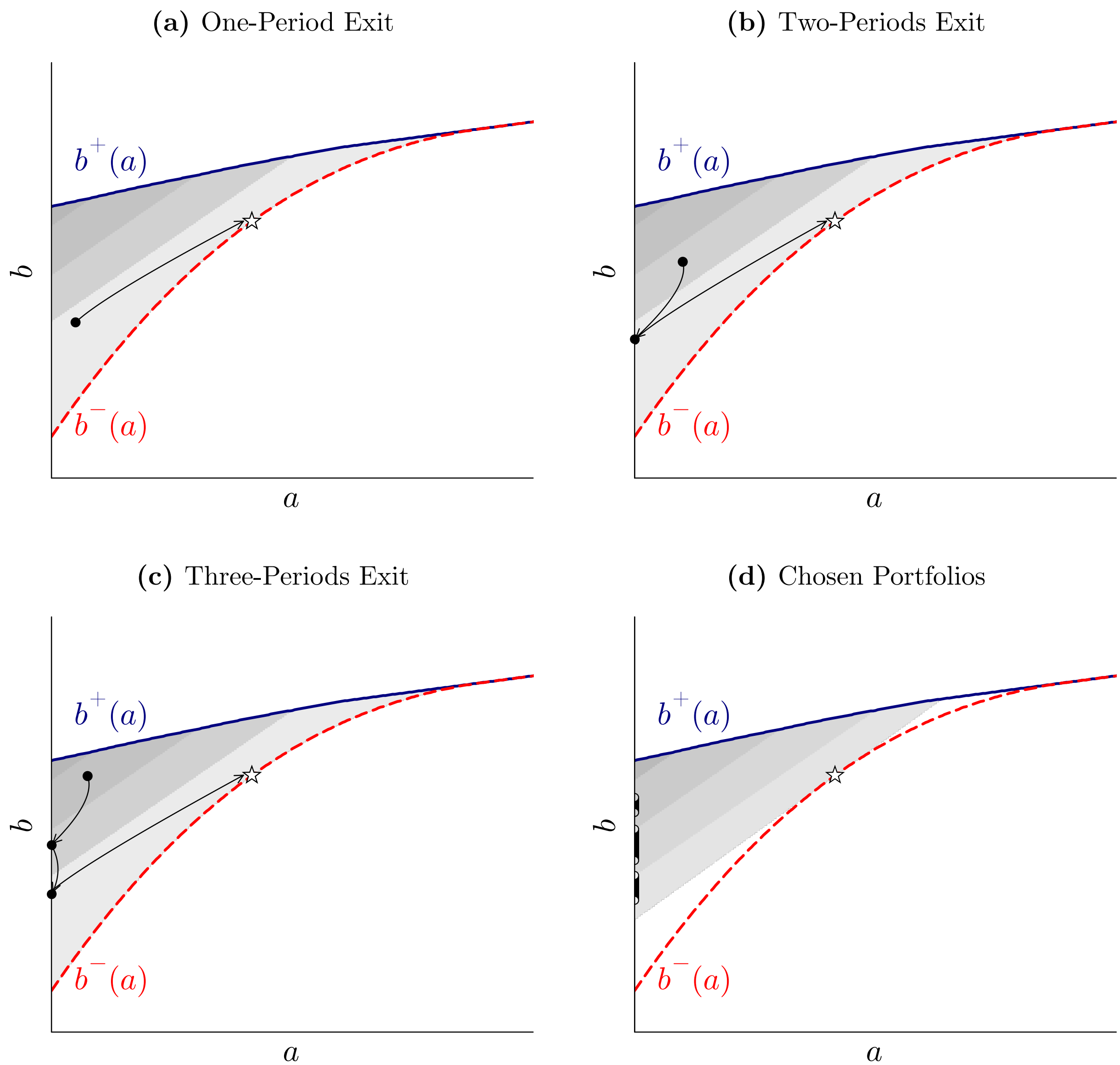

Figure 3: Periods to exit and portfolios. The figure presents examples where the government exits in one period (panel a), two periods (panel b), and three periods (panel c). Panel (d) shows all the possible portfolios the government chooses for any initial portfolio in the crisis zone such that $a - b < a^\star - b^\star$. The star point in all four panels indicate the portfolio $(a^\star, b^\star)$.

Policy correspondences. To further inspect the portfolio of the government, Figure 4 presents the policies for reserves, borrowing, NFA, and consumption in panels (a),(b),(c), and (d), respectively. These policies correspond to the solution of the government problem (5) (i.e., they are the policies conditional on repayment and having access to the bond market). The plots are presented for a range of initial values for b and for $a = 0$. The grey areas denote correspondences.

The figure highlights three vertical lines: the first vertical line denotes $b^{-}(0)$, the middle vertical line denotes $b^{-}(a^{\star}) - a^{\star}$, and the third vertical line denotes $b^{+}(0)$. We note that the middle vertical line corresponds to a portfolio with zero initial reserves and with the same NFA as the lowest-NFA safe portfolio. In terms of Figure 1, this corresponds to the solid dot where the arrow originates.

Let us start from the first vertical line, which indicates the boundary between the safe zone and crisis zone for zero reserves. For debt levels to the left of this line, the government is already in the safe zone and there is a range of portfolios that deliver the optimum solution, as illustrated in the shaded area in panels (a) and (b). Consumption and the NFA, however, are uniquely determined (panels [c] and [d]). In particular, consumption is given by $c = y - (1 - \beta)b$ and the NFA equals $-b$.

For debt levels between the first and the middle vertical line, the government is initially in the crisis zone. Because the government starts from an NFA position that is higher than the lowest-NFA safe portfolio, it can exit the crisis zone without cutting consumption, as implied by Proposition 6. As in the previous case, portfolios are correspondences and the NFA is kept constant. However, in this region, *the government must accumulate a strictly positive amount of reserves to exit the crisis zone.* Moreover, it finances the accumulation of reserves with new debt issuances.

For debt levels immediately to the right of the middle vertical line, the government still exits in one period, but it now needs to increase the net foreign asset position to reach the safe zone. Consistent with Proposition 5, the government finds it strictly optimal to choose $(a^{\star}, b^{\star})$. One can also see that as the initial debt level increases further, there is a sharper drop in consumption (panel [d]). This is intuitive because while the initial debt level is increased, the government continues to choose the portfolio $(a^{\star}, b^{\star})$. It is also interesting to note that the policy function for debt lies above the 45-degree line (panel [b]). That is, *the government increases the amount of debt to exit the crisis zone.*

As we move to the right (still between the middle and the third vertical lines), borrowing falls discretely and the government chooses zero reserves. At this point, the government now takes two periods to exit. We can also see that the policy for b' lies below the 45° line. That

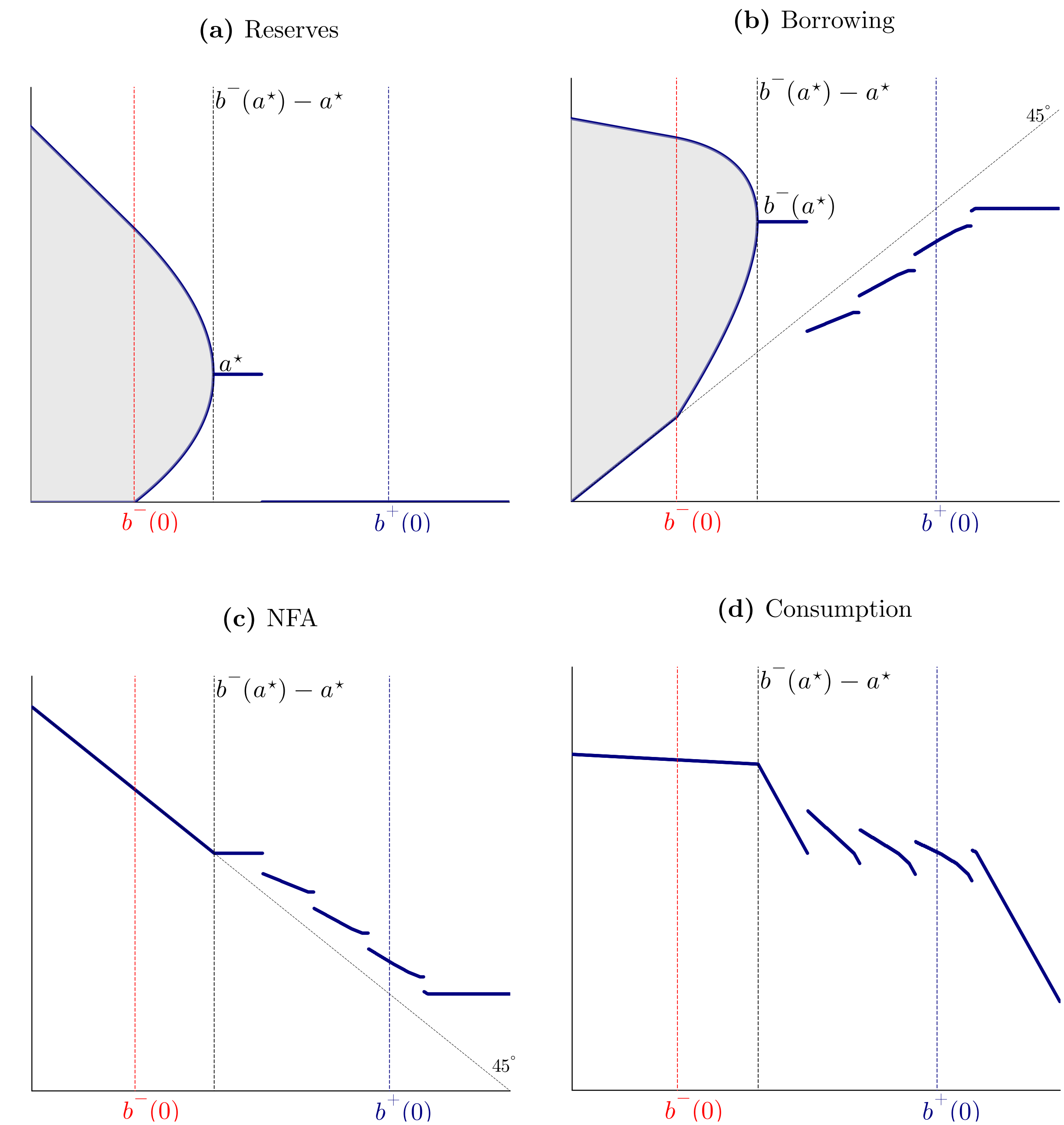

Figure 4: Policy correspondences. The plots are for an initial level of reserves $a = 0$ and a range of values for initial debt (in the horizontal axis). The top panels denote $a'(0, b)$ and $b'(0, b)$ and the bottom panels denote $a'(0, b) - b'(0, b)$, and $c(0, b)$. Light grey areas represent the portfolio indeterminacy in that region.

is, the government reduces its debt and delays the accumulation of reserves. As we move further to the right, we see another point of discontinuity when the government postpones by one more period the planned exit of the crisis zone. Specifically, at the point of discontinuity, the government is indifferent between choosing a certain level of borrowing and a significantly higher one that delays the exit time by one period. When the government chooses to postpone by one period the exit, we can see that consumption jumps upward as we increase the initial debt (and then falls continuously with the initial debt level).[18] Once debt exceeds the third vertical line, denoting $b^{+}(0)$, the government chooses to default in equilibrium.[19]

To conclude, we have characterized the optimal strategy for a government that seeks to exit the crisis zone. The key takeaway is that when the government is deep in the crisis zone, the optimal policy is to reduce the debt and keep zero reserves. As the government approaches the safe zone, it is optimal to increase borrowing and accumulate reserves.

3.4 Quantitative Results

In this section, we present a calibration of the model to assess the quantitative role of reserves. We use data from Italy to calibrate the model. Parameter values are presented in Table 1.[20]

Table 1: Parameter values

Parameter	Value	Description	Source
y	1	Endowment	Normalization
σ	2	Risk-aversion	Standard
r	3%	Risk-free rate	Standard
$1/\delta$	6	Maturity of debt	Italian Debt
β	0.97	Discount factor	$\beta(1+r)=1$
$\underline{c}$	0.68	Consumption floor	Bocola and Dovis (2019)
λ	0.5%	Sunspot probability	Baseline
ϕ	0.33	Default Cost	Debt-to-income =100%

[18]The saw-tooth pattern for NFA and consumption in panels (c) and (d) respectively, have some common features with Cole and Kehoe (2000)'s model with one-period debt and no assets. One subtle difference with long-term debt is that the policy function for NFA has several flat regions. This is related to the debt dilution effect of long-term debt which implies that the government is deterred from a slight increase in borrowing despite the higher initial stock of debt. This is because higher borrowing would increase the exit period and discretely reduce the bond price. Notice that at these flat portions, consumption experiences a sharper drop.

[19]As mentioned above, the figure presents the policy conditional on repayment. Notice also that in the figure borrowing never exceeds $b^{+}(0)$ (because the government would face a zero bond price in that case).

[20]The algorithm is described in Appendix C.

A model period is one year and income is normalized to one. As in Conesa and Kehoe (2017) and Bocola and Dovis (2019), we assume the utility function takes the form

$$u(c) = \frac{(c - \underline{c})^{1-\sigma}}{1-\sigma},$$

where $\underline{c}$ stands for the level of consumption that cannot be changed in the short run. Following Bocola and Dovis (2019), we set $\underline{c} = 0.68$, based on the measure of non-discretionary spending for the Italian government. We set $\sigma = 2$ and $r = 0.03$, which are common values in the literature.

We calibrate the cost of default ϕ so that the midpoint between $b^{-}(0)$ and $b^{+}(0)$ is roughly 100% of GDP, the level of Italy's debt in the run-up to the sovereign debt crisis in 2012. The maturity of the debt is set to 6 years, which implies $\delta = 1/6$. Finally, we use $\lambda = 0.5\%$ as a baseline value. This value is important for the speed at which the economy exits the crisis zone but does not affect the lowest-NFA safe portfolio $(a^{\star}, b^{\star})$.

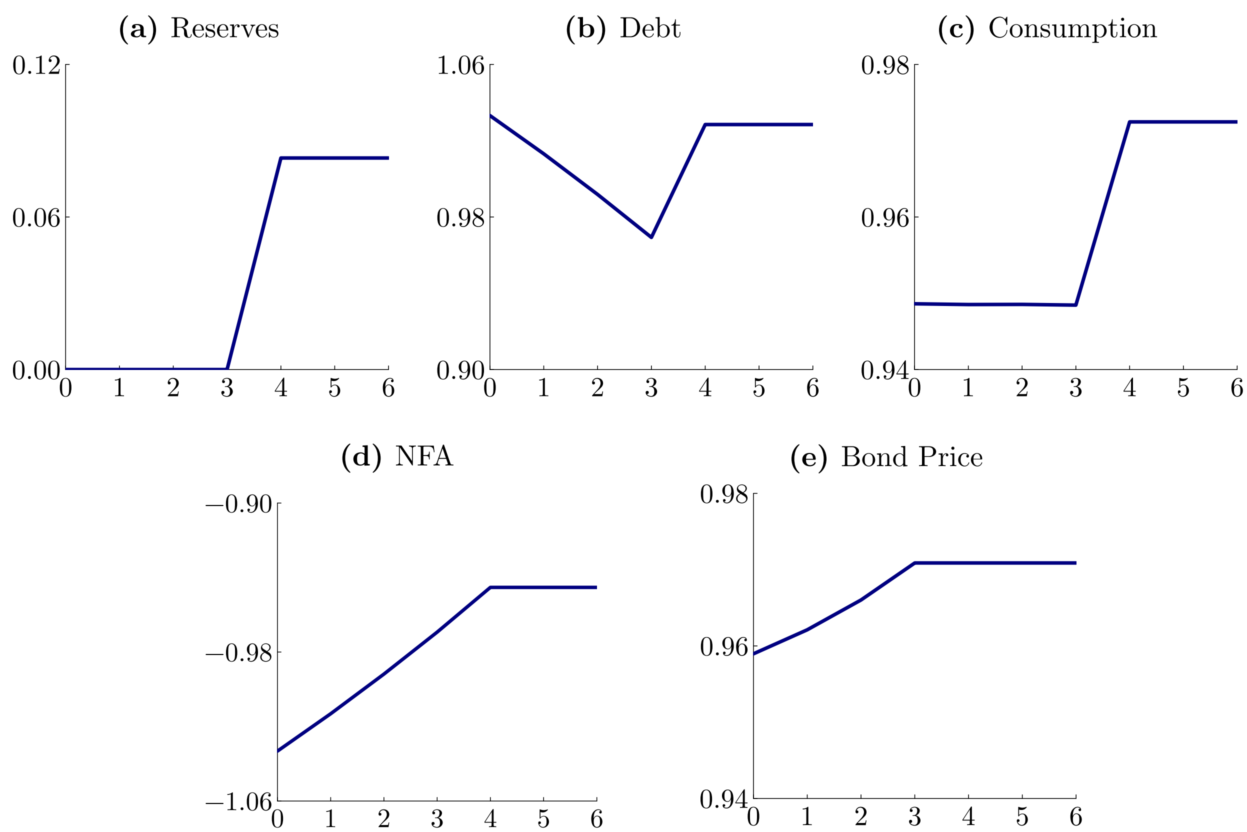

Figure 5: Deleveraging dynamics. The government is assumed to start in the crisis zone with $b = 1.04$ and zero initial reserves. Panels (a), (b), and (d) plot beginning-of-period levels of reserves, debt, and NFA, respectively.

Simulation results. We obtain that the lowest-NFA safe portfolio is $a^\star = 0.08, b^\star = 1.04$. Recall that we normalize income to one, so the values can be interpreted as fractions of GDP. Figure 5 shows a time series simulation for a government that starts in the crisis zone. For the initial portfolio at $t = 0$ considered, the government reaches the safe zone in four periods. For $t \geq 4$, all variables therefore remain constant.

As can be observed in panels (a) and (d), the NFA increases monotonically while the debt issuances fall initially and then increase at $t = 3$ upon exiting the crisis zone. In addition, consumption (panel [c]) is weakly increasing over time, as shown in Proposition 1. In particular, consumption is constant for $t = 0, 1, 2, 3$ and increases at $t = 4$ once the economy reaches the safe zone. Finally, panel (e) illustrates how the bond price increases monotonically over time. As highlighted in the recursion (14), as the government approaches the safe zone, the probability of a future default falls and the bond price increases.

Sensitivity. In Figure 6, we examine how changes in the parameters for the debt duration and risk aversion alter the optimal portfolio for the government. Specifically, we vary δ and σ and recalibrate ϕ to match the same $b^-(0)$ as in the baseline calibration. The figure presents the value of $a^\star$.

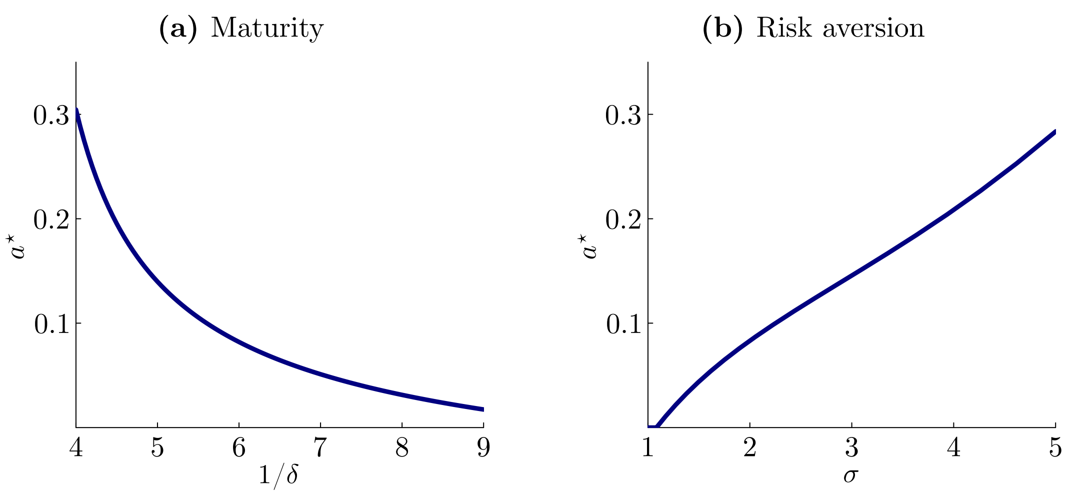

Figure 6: Sensitivity analysis. The panels show the level of reserves $a^\star$ for different parameter values for δ and σ. In the simulations, the value of ϕ is recalibrated to match the same debt level $b^-(0)$ as in the baseline calibration.

Panel (a) shows that when the debt maturity becomes shorter, the government accumulates a larger amount of reserves. A shorter maturity implies that a larger fraction of the debt becomes due each period.[21] Therefore, the government needs a higher amount of reserves to be safe from a rollover crisis. For a maturity of 4 years, the stock of reserves can reach close to 30% of GDP. Panel (b) shows that for higher risk aversion, the government accumulates a larger amount of reserves. A higher risk aversion implies higher curvature in the utility function. Thus, reserves provide a higher marginal value when the government faces a run.

Counterfactual without reserves. We next compare our benchmark economy with an economy where the government never accumulates reserves under repayment. Namely, the government starts with portfolio (a, b) at some t, and from $t+1$ onward, it is forced to set $a' = 0$.[22]

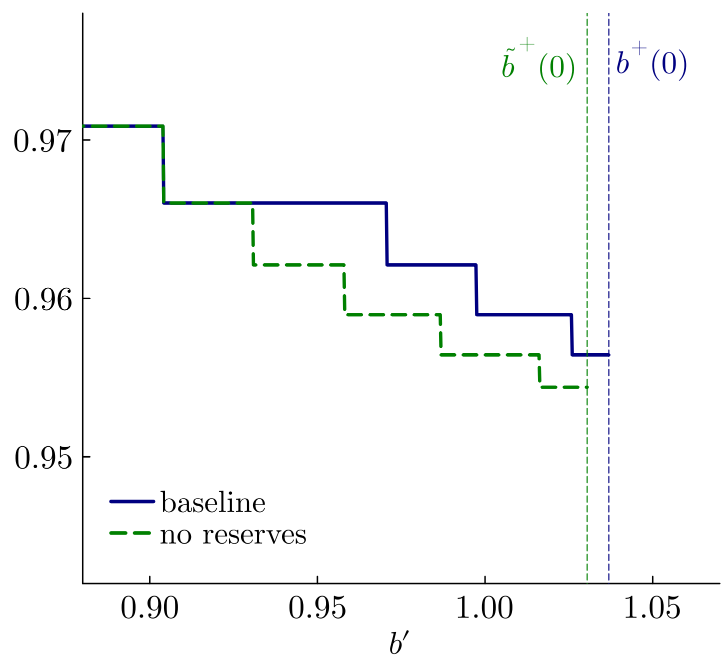

Figure 7: Bond prices schedules with and without reserves. The blue-solid line denotes the bond price schedule in the baseline model at zero reserves, $q(0, b')$. The green-dashed line denotes the bond price schedule in the economy without reserves, $\tilde{q}(b')$. The green vertical line $\tilde{b}^{+}(0)$ is given by $\tilde{V}_R^{+}(\tilde{b}^{+}(0)) = V_D(0)$, where $\tilde{V}_R^{+}$ corresponds to the value function when the government cannot accumulate reserves under repayment in the future.

[21] Recall, however, that in the limit when the maturity is one period there is no scope for reserve accumulation, as established in Proposition 4.

[22] We impose the restriction to never accumulate reserves only under repayment, so this means that V_D remains the same as in the baseline.

Figure 7 compares the bond price schedules with and without reserves. One can see that the bond price with reserves is always above the economy without reserves, strictly so for a range of values of debt. In the absence of reserves, exiting the crisis zone becomes more costly, as now the government's terminal point is $(0, b^{-}(0))$ instead of $(a^{\star}, b^{\star})$. That is, the government must either cut consumption more through the deleveraging process or take more time to reach the safe zone. Because exiting is more costly, it follows that the debt threshold b^{+} is reduced. The lack of access to reserves makes debt less sustainable.[23]

4 Conclusions

This paper considers a government subject to the risk of a rollover crisis and asks, should the government accumulate reserves or reduce debt to make itself less vulnerable? The theory shows that the optimal policy entails initially reducing debt, followed by a subsequent increase in both debt and reserves. That is, a highly indebted government should first raise its net foreign asset position by lowering debt and abstaining from holding reserves. Once the government approaches the safe zone, it becomes optimal to issue more debt to accumulate reserves. Furthermore, the theory reveals how issuing debt to accumulate reserves can lower spreads by reducing the vulnerability to a rollover crisis.

Our findings speak to central bank policy discussions on the appropriate level of international reserves (e.g., IMF, 2016, 2023). Following a debt crisis, the IMF often prescribes that countries accumulate reserves to improve their liquidity position. Our results show, however, that holding reserves is not optimal at the beginning of a deleveraging process. Rather, a highly indebted government should first reduce debt and postpone the accumulation of reserves until the increase in reserves makes the government safe from a rollover crisis.

[23]Appendix B presents more details. Figure A1 presents the policy function for NFA and consumption without reserves with the same baseline parameters. Figure A2 presents a simulation where the government without reserves takes more time to exit the crisis zone and chooses lower consumption during this transition as it deleverages.

References

Aguiar, Mark and Gita Gopinath, "Defaultable Debt, Interest Rates and the Current Account," *Journal of International Economics*, 2006, *69* (1), 64–83.

Aguiar, Mark and Manuel Amador, "Take the Short Route: How to Repay and Restructure Sovereign Debt with Multiple Maturities," 2013. NBER Working Paper 19717.

Aguiar, Mark and Manuel Amador, "Sovereign Debt," in Gita Gopinath, Elhanan Helpman, and Kenneth Rogoff, eds., *Handbook of International Economics*, Vol. 4, North Holland, 2014, pp. 647–687.

Aguiar, Mark and Manuel Amador, *The Economics of Sovereign Debt and Default*, Princeton University Press, 2023.

Aguiar, Mark and Manuel Amador, "Sovereign swaps and sovereign default: Fundamental versus confidence risk," 2024. NBER Working Paper 19717.

Aguiar, Mark, Manuel Amador, Emmanuel Farhi, and Gita Gopinath, "Crisis and commitment: Inflation credibility and the vulnerability to sovereign debt crises," Technical Report, NBER Working Paper No. 19516 2013.

Aguiar, Mark, Manuel Amador, Emmanuel Farhi, and Gita Gopinath, "Coordination and crisis in monetary unions," *The Quarterly Journal of Economics*, 2015, *130* (4), 1727–1779.

Aguiar, Mark, Satyajit Chatterjee, Harold Cole, and Zachary Stangebye, "Quantitative Models of Sovereign Debt Crises," in John B. Taylor and Harald Uhlig, eds., *Handbook of Macroeconomics*, Vol. 2, Elsevier, 2016, pp. 1697–1755.

Aguiar, Mark, Satyajit Chatterjee, Harold L Cole, and Zachary Stangebye, "Self-Fulfilling Debt Crises, Revisited: The Art of the Desperate Deal," *Journal of Political Economy*, 2022, *130* (5), 1147–1183.

Alfaro, Laura and Fabio Kanczuk, "Optimal Reserve Management and Sovereign Debt," *Journal of International Economics*, 2009, *77* (1), 23–36.

Angeletos, George-Marios, "Fiscal policy with noncontingent debt and the optimal maturity structure," *Quarterly Journal of Economics*, 2002, *117 (3)*, 1105–1131.

Araujo, Aloisio, Marcia Leon, and Rafael Santos, "Welfare analysis of currency regimes with defaultable debts," *Journal of International Economics*, 2013, *89* (1), 143–153.

Arellano, Cristina, "Default Risk and Income Fluctuations in Emerging Economies," *American Economic Review*, 2008, *98* (3), 690–712.

Arellano, Cristina and Ananth Ramanarayanan, "Default and the Maturity Structure in Sovereign Bonds," *Journal of Political Economy*, 2012, *120* (2), 187–232.

Ayres, Joao, Gaston Navarro, Juan Pablo Nicolini, and Pedro Teles, "Sovereign default: The role of expectations," *Journal of Economic Theory*, 2018, *175*, 803–812.

Ayres, Joao, Gaston Navarro, Juan Pablo Nicolini, and Pedro Teles, "Self-fulfilling debt crises with long stagnations," 2023. Minneapolis Fed Working Paper.

Bacchetta, Philippe, Elena Perazzi, and Eric Van Wincoop, "Self-fulfilling debt crises: What can monetary policy do?," *Journal of International Economics*, 2018, *110*, 119–134.

Barro, Robert J, "Notes on optimal debt management," *Journal of Applied Economics*, 1999, *2* (2), 281–289.

Barro, Robert J, "Optimal management of indexed and nominal debt," *Annals of Economics and Finance*, 2003, *4*, 1–15.

Bassetto, Marco and Carlo Galli, "Is inflation default? The role of information in debt crises," *American Economic Review*, 2019, *109* (10), 3556–3584.

Bhandari, Anmol, David Evans, Mikhail Golosov, and Thomas J Sargent, "Fiscal policy and debt management with incomplete markets," *The Quarterly Journal of Economics*, 2017, *132* (2), 617–663.

Bianchi, Javier and César Sosa-Padilla, "Reserve Accumulation, Macroeconomic Stabilization, and Sovereign Risk," *The Review of Economic Studies*, 07 2023.

Bianchi, Javier and Jorge Mondragon, "Monetary independence and rollover crises," *The Quarterly Journal of Economics*, 2022, *137* (1), 435–491.

Bianchi, Javier, Juan Carlos Hatchondo, and Leonardo Martinez, "International reserves and rollover risk," *American Economic Review*, 2018, *108* (9), 2629–2670.

Bianchi, Javier, Pablo Ottonello, and Ignacio Presno, "Fiscal stimulus under sovereign risk," *Journal of Political Economy*, 2023, *131* (9), 2328–2369.

Bocola, Luigi and Alessandro Dovis, "Self-Fulfilling Debt Crises: A Quantitative Analysis," *American Economic Review*, 2019, *109* (12), 4343–4377.

Buera, Francisco and Juan Pablo Nicolini, "Optimal maturity of government debt without state contingent bonds," *Journal of Monetary Economics*, 2004, *51* (3), 531–554.

Bulow, Jeremy and Kenneth Rogoff, "Sovereign Debt: Is to Forgive to Forget?," *The American Economic Review*, 1989, *79* (1), 43–50.

Camous, Antoine and Russell Cooper, ""Whatever it takes" is all you need: Monetary policy and debt fragility," *American Economic Journal: Macroeconomics*, 2019, *11* (4), 38–81.

Chatterjee, Satyajit and Burcu Eyigungor, "Maturity, indebtedness, and default risk," *American Economic Review*, 2012, *102* (6), 2674–2699.

Cole, Harold L. and Timothy J. Kehoe, "Self-Fulfilling Debt Crises," *Review of Economic Studies*, 2000, *67* (1), 91–116.

Cole, Harold L, Daniel Neuhann, and Guillermo Ordonez, "Information Spillovers and Sovereign Debt: Theory Meets the Eurozone Crisis," 2024. Forthcoming, Review of Economic Studies.

Conesa, Juan Carlos and Timothy J Kehoe, "Gambling for Redemption and Self-Fulfilling Debt Crises," *Economic Theory*, 2017, *64* (4), 707–740.

Conesa, Juan Carlos and Timothy J Kehoe, "Preemptive Austerity with Rollover Risk," *Forthcoming, Journal of International Economcis*, 2024.

Corsetti, Giancarlo and Fred Seunghyun Maeng, "Debt crises, fast and slow," *Journal of the European Economic Association*, 2023.

Corsetti, Giancarlo and Luca Dedola, "The mystery of the printing press: Monetary policy and self-fulfilling debt crises," *Journal of the European Economic Association*, 2016, *14* (6), 1329–1371.

Corsetti, Giancarlo and Seung Hyun Maeng, "The Theory of Reserve Accumulation, Revisited," 2024. Mimeo, European University Institute.

Eaton, Jonathan and Mark Gersovitz, "Debt with Potential Repudiation: Theoretical and Empirical Analysis," *Review of Economic Studies*, 1981, *48 (2)*, 289–309.

Hatchondo, Juan Carlos and Leonardo Martinez, "Long-Duration Bonds and Sovereign Defaults," *Journal of International Economics*, 2009, *79 (1)*, 117–125.

Hatchondo, Juan Carlos, Leonardo Martinez, and César Sosa-Padilla, "Debt dilution and sovereign default risk," *Journal of Political Economy*, 2016, *124 (5)*, 1383–1422.

Hernandez, Juan, "How International Reserves Reduce The Probability of Debt Crises," 2018. Inter-American Development Bank Discussion Paper IDB-DP-579.

IMF, "Assessing reserve adequacy," 2016. Guidance Note to Staff on Assessing Reserve Adequacy and Related Issues.

IMF, "Assessing reserve adequacy," 2023. IMF Note.

Lorenzoni, Guido and Ivan Werning, "Slow moving debt crises," *American Economic Review*, 2019, *109* (9), 3229–3263.

A Proofs

A.1 Proof of Proposition 1

Proof. Toward a contradiction, suppose that $c_t > c_{t+1}$ in the crisis zone is optimal. Consider a policy at t such that consumption $\tilde{c}_t = c_t - \Delta$ with $\Delta > 0$ and a reserve policy $\tilde{a} = a_{t+1}$. In addition, assume that the government keeps the same (a_{t+2}, b_{t+2}) as the original allocation in the event that $\zeta_{t+1} = 0$. Notice that this policy is feasible and implies that continuation values at $t+2$ remain the same as in the original allocation. Without loss of generality, suppose the alternative portfolio keeps the government in the crisis zone (i.e., $(\tilde{a}, \tilde{b}) \in \mathbf{C}$). We then have that $q(a_{t+1}, b_{t+1}) = q(\tilde{a}, \tilde{b})$, and from the budget constraint,

$$\tilde{b} = b_{t+1} - \frac{\Delta}{q(\tilde{a}, \tilde{b})}. \tag{A.1}$$

We will argue that welfare increases with this deviation. That is, given the same continuation value $t+2$, this means that the sum of utility from period t and expected utility from $t+1$ consumption is higher under this deviation. That is, we want to show that

$$\begin{aligned} &u(c_t - \Delta) + \beta(1-\lambda)u\Big(y + a_{t+1} - \frac{a_{t+2}}{1+r} - \kappa\tilde{b} + q(a_{t+2}, b_{t+2})\left[b_{t+2} - (1-\delta)\tilde{b}\right]\Big) > \\ &\quad u(c_t) + \beta(1-\lambda)u\Big(y + a_{t+1} - \frac{a_{t+2}}{1+r} - \kappa b_{t+1} + q(a_{t+2}, b_{t+2})\left[b_{t+2} - (1-\delta)b_{t+1}\right]\Big), \end{aligned} \tag{A.2}$$

where $\kappa \equiv \frac{\delta+r}{1+r}$. The right-hand side of (A.2) can be be simply written as:

$$u(c_t) + \beta(1-\lambda)u(c_{t+1}). \tag{A.3}$$

Using (A.1) and the price equation (4), we can write the left-hand side of (A.2) as:

$$u(c_t-\Delta)+\beta(1-\lambda)u\left(y+a_{t+1}-\frac{a_{t+2}}{1+r}-\kappa b_{t+1}+q(a_{t+2}, b_{t+2})\left[b_{t+2}-(1-\delta)b_{t+1}\right]+\frac{\Delta(1+r)}{(1-\lambda)}\right).$$

Using (A.3) we can simplify the left-hand side of (A.2) as:

$$u(c_t - \Delta) + \beta(1-\lambda)u\left(c_{t+1} + \frac{\Delta(1+r)}{(1-\lambda)}\right) \tag{A.4}$$

Substituting (A.3)-(A.4) in (A.2), we arrive at

$$u(c_t-\Delta)+\beta(1-\lambda)u\left(c_{t+1}+\frac{\Delta(1+r)}{1-\lambda}\right)>u(c_t)+\beta(1-\lambda)u(c_{t+1}),$$

which holds if and only if

$$u(c_t-\Delta)-u(c_t)+\beta(1-\lambda)\left[u\left(c_{t+1}+\frac{\Delta(1+r)}{1-\lambda}\right)-u(c_{t+1})\right]>0\,.$$

Dividing both sides by Δ, using $\beta(1+r)=1$ and re-arranging terms we get

$$\frac{u\left(c_{t+1}+\frac{\Delta(1+r)}{1-\lambda}\right)-u(c_{t+1})}{\frac{\Delta(1+r)}{1-\lambda}}-\frac{u(c_t)-u(c_t-\Delta)}{\Delta}>0\,.$$

For small Δ, it follows that we can improve utility, reaching a contradiction:

$$\lim_{\Delta\to 0}\frac{u\left(c_{t+1}+\frac{\Delta(1+r)}{1-\lambda}\right)-u(c_{t+1})}{\frac{\Delta(1+r)}{1-\lambda}}-\lim_{\Delta\to 0}\frac{u(c_t)-u(c_t-\Delta)}{\Delta}\;=\;u'(c_{t+1})\;-\;u'(c_t)\;>\;0\,.$$

□

A.2 Proof of Lemma 1

Proof. Part (i):

Consider problem (15). A necessary and sufficient condition for optimality implies

$$u'\left(y+a-\frac{a'}{1+r}-\left(\frac{\delta+r}{1+r}\right)b\right)\geq\beta(1+r)u'\left(y+(1-\beta)(a'-(1-\delta)b)\right), \tag{A.5}$$

with equality if $a'>0$. In the case of equality (and using that $\beta(1+r)=1$), we have

$$y+a-\frac{a'}{1+r}-\left(\frac{\delta+r}{1+r}\right)b=y+(1-\beta)(a'-(1-\delta)b),$$

which implies

$$a'=a-\delta b.$$

The solution is therefore

$$a'=\max[0,a-\delta b]. \tag{A.6}$$

That is, if $a > \delta b$, then the unconstrained solution is feasible. Otherwise, $a' = 0$.

$\square$

Proof. Part (ii): Consider the problem when there is no run. In that case, $b' = (1-\delta)b$, and $a' = a - \delta b$, and $c = y + (1-\beta)(a-b)$ is feasible because $a \geq \delta b$. Moreover, the continuation value is safe by the assumption that $(a - \delta b, (1-\delta)b) \in \mathbf{S}$. Thus, the government can achieve the ideal stationary outcome. Consider now the problem under a run (7). Given that $a' = 0$ and $b' = b(1-\delta)$ remains feasible, the government under a run can achieve the same value as the government that does not face a run. We thus obtain the result that $V_R^+ = V_R^-$, as we wanted to show. $\square$

A.3 Proof of Proposition 2

We first prove the following lemma:

Lemma A.1. *For any $(a, b) \in \mathbf{C}$, we have that $c_R^-(a, b) < c_R^+(a, b) < c_D(a)$.*

Proof. By Proposition 1, consumption is increasing over time in the crisis zone. Moreover, $c_R^+(a, b^+(a)) > c_D$ for the government to be indifferent between repaying and defaulting at the portfolio $(a, b^+(a))$. Therefore it must be that $c_R^+(a, b) < c_D(a)$ for $(a, b) \in \mathbf{C}$.

The result that $c_R^-(a, b) < c_D(a)$ is as follows. Let $\bar{a}(a, b)$ be the solution of (15). If $(\bar{a}(a, b), (1-\delta)b) \in \mathbf{S}$, the result that $c_R^-(a, b) < c_D(a)$ is immediate from replacing (A.6) in the government budget constraint.

If $(\bar{a}(a, b), (1-\delta)b) \notin \mathbf{S}$, the government has a higher incentive to save in reserves and thus consumption is lower. To see this, first note that if $(a', b') \in \mathbf{C}$, we have

$$u'\left(y + a - \frac{a'}{1+r} - \left(\frac{\delta + r}{1+r}\right)b\right) \geq \beta(1+r)[\lambda u'(c_D(a')) + (1-\lambda)u'\left(c^+(a', b(1-\delta)\right). \quad \text{(A.7)}$$

If $c_D(a) < c_R^-(a, b)$, we have $a' < a - (\delta + r)b$, and thus the end of period net foreign asset position is $a' - b' = a' - b(1-\delta) < a - (\delta + r)b - b(1-\delta) = a - (1+r)b < a - b$, which in turn implies $a' < a$ and $c_D(a) > c_D(a')$.

Toward a contradiction, suppose that $c_D(a) < c_R^-(a, b)$. Using (A.7) and the fact that $c_R^+(a, b) < c_D(a)$, we arrive at

$$u'\left(c_D(a)\right) > u'\left(c_D(a')\right),$$

which implies $c_D(a) < c_D(a')$ and thus a contradiction.

$\square$

We now prove Proposition 2.

Proof. We have that $V_D(a) = V_R^-(a, b^-(a))$. Differentiating with respect to a, we obtain

$$\frac{\partial V_D(a)}{\partial a} = \frac{\partial V_R^-(a, b^-(a))}{\partial a} + \frac{\partial V_R^-(a, b^-(a))}{\partial b} \frac{\partial b^-}{\partial a}.$$

Rearranging, we get

$$\frac{\partial b^-}{\partial a} = \frac{\frac{\partial V_D(a)}{\partial a} - \frac{\partial V_R^-\left(a, b^-(a)\right)}{\partial a}}{\frac{\partial V_R^-(a, b^-(a))}{\partial b}}. \tag{A.8}$$

Proceeding analogously for $b^+(a)$, we have that

$$\frac{\partial b^+}{\partial a} = \frac{\frac{\partial V_D(a)}{\partial a} - \frac{\partial V_R^+\left(a, b^+(a)\right)}{\partial a}}{\frac{\partial V_R^+(a, b^+(a))}{\partial b}}. \tag{A.9}$$

Applying the envelope condition in (5), (7), and (12), we obtain

$$\frac{\partial V_R^-(a, b)}{\partial a} = u'\left(c_R^-(a, b)\right), \tag{A.10}$$

$$\frac{\partial V_D(a)}{\partial a} = u'\left(c_D(a)\right), \tag{A.11}$$

$$\frac{\partial V_R^+(a, b)}{\partial a} = u'\left(c_R^+(a, b)\right). \tag{A.12}$$

From Lemma A.1, it then follows that $\frac{\partial V_R^-(a,b)}{\partial a} > \frac{\partial V_R^+(a,b)}{\partial a} > \frac{\partial V_D(a)}{\partial a}$ for any $(a, b) \in \mathbf{C}$. Using these inequalities in (A.8) and (A.9), we therefore have $\frac{\partial b^-}{\partial a} > 0$ and $\frac{\partial b^+}{\partial a} > 0$, implying that $b^-(a)$ and $b^+(a)$ are increasing in a. Moreover, using $\frac{\partial V_R^-(a,b)}{\partial a} > \frac{\partial V_R^+(a,b)}{\partial a}$ for $(a, b) \in \mathbf{C}$, the result that $\frac{\partial b^-}{\partial a} > \frac{\partial b^+}{\partial a} > 0$ for any a such that $b^-(a) < b^+(a)$ follows.

$\square$

A.4 Proof of Proposition 3

Proof. Toward a contradiction, suppose that the lowest-NFA safe portfolio $(a^\star, b^\star)$ is such that $a^\star = 0$. Consider an alternative portfolio $(\tilde{a}, b^-(\tilde{a}))$, with $\tilde{a} > 0$. By continuity of $\frac{\partial b^-(a)}{\partial a}$, we have

$$b^{-}(\tilde{a}) - b^{-}(0) = \int_0^{\tilde{a}} \frac{\partial b^{-}(a)}{\partial a} da > \int_0^{\tilde{a}} da = \tilde{a}. \tag{A.13}$$

The argument for the strict inequality comes from showing that $\frac{\partial b^{-}(a)}{a} > 1$ in the interval $[0, \tilde{a}]$. To prove this, we first note that the envelope condition in (7) is

$$\frac{\partial V_R^{-}(a,b)}{\partial b} = -u'\left(y + a - \left(\frac{\delta + r}{1+r}\right) b - \frac{a'}{1+r}\right)\left(\frac{\delta + r}{1+r}\right) - \beta(1-\delta)\mathbb{E}\frac{\partial V(a', b(1-\delta), \zeta')}{\partial b}. \tag{A.14}$$

If we evaluate the above expression at $(0, b^{-}(0))$, then the government is repaying today, and it is straightforward to see that it will be safe tomorrow since it will have less debt and cannot have fewer reserves: $b' = (1-\delta)b$ and $a' \geq 0$. Moreover, we can show that a government that starts the period in the safe region will always choose to stay in the safe region. Therefore, the envelope condition above, evaluated at $(0, b^{-}(0))$ can be rewritten as:

$$\frac{\partial V_R^{-}(0, b^{-}(0))}{\partial b} = -u'\left(y - \left(\frac{\delta + r}{1+r}\right) b^{-}(0)\right)\left(\frac{\delta + r}{1+r}\right) - \beta(1-\delta)u\left(y - (1-\beta)(1-\delta)b^{-}(0)\right) \tag{A.15}$$

where we have used Lemma 1 to have $a' = 0$. We can then replace (A.15), together with (A.10), and (A.11) into (A.8) to obtain

$$\left.\frac{\partial b^{-}(a)}{\partial a}\right|_{a=0} = \frac{u'\left(y - \left(\frac{\delta+r}{1+r}\right) b^{-}(0)\right) - u'(y)}{u'\left(y - \left(\frac{\delta+r}{1+r}\right) b^{-}(0)\right)\left(\frac{\delta+r}{1+r}\right) + \beta(1-\delta)u\left(y - (1-\beta)(1-\delta)b^{-}(0)\right)}. \tag{A.16}$$

Condition (18) guarantees that $\left.\frac{\partial b^{-}(a)}{\partial a}\right|_{a=0} > 1$. By continuity, we then have that for a small $\tilde{a}$, it must be that $\frac{\partial b^{-}(\tilde{a})}{\partial a} > 1$.

Rearranging (A.13), we arrive at

$$\tilde{a} - b^{-}(\tilde{a}) < 0 - b^{-}(0),$$

which implies that we were able to find a portfolio $(\tilde{a}, b^{-}(\tilde{a})) \in \mathbf{S}$ with a lower net foreign asset position. We thus find a contradiction that the original portfolio with $a^{\star} = 0$ was part of a lowest-NFA safe portfolio. □

A.5 Proof of Proposition 4

Proof. With $\delta = 1$, the value of repayment in a run becomes

$$V_R^-(a,b) = \max_{a' \geq 0} u\left(y + a - b - \frac{a'}{1+r}\right) + \beta V_S(a'), \tag{A.17}$$

where we have used that the government is safe tomorrow after repaying all the debt. The solution for a' is given by (A.6). Under the assumption that $a < b$—and thus a negative net foreign asset position which we will verify below—we have $a' = 0$. Replacing the values of V_S and V_D from (11) and (12), we have that the government is indifferent between repaying in a run and defaulting if the following holds:

$$u(y + a - b) = \frac{u(y + a(1-\beta)) - \phi}{1-\beta} - \frac{\beta}{1-\beta} u(y). \tag{A.18}$$

The lowest-NFA safe portfolio solves

$$\min_{a \geq 0, b} \; a - b,$$
$$\text{subject to}$$
$$u(y + a - b) + \frac{\beta}{1-\beta} u(y) \geq \frac{u(y + a(1-\beta)) - \phi}{1-\beta}.$$

Optimality with respect to a and b implies that

$$\begin{aligned}
&1 \geq \lambda[u'(y + a - b) - u'(y + a(1-\beta)) \quad \text{with equality if } a > 0, \\
&1 = \lambda u'(y + a - b).
\end{aligned}$$

Combining these two conditions, we get

$$u'(y + a - b) \geq u'(y + a - b) - u'(y + a(1-\beta)).$$

Given that $u'(y + a(1-\beta)) > 0$, this condition holds with strict inequality, thus implying that $a^\star = 0$.

With $\delta = 0$, the value of repaying in a run becomes $V_R^-(a,b) = u(y + (a-b)(1-\beta))/(1-\beta)$

and lowest-NFA safe portfolio solves

$$\min_{a \geq 0, b} \; a - b,$$

$$\text{subject to}$$

$$u(y + (a-b)(1-\beta)) \geq \; u(y + a(1-\beta)) - \phi.$$

Notice the constraint gets tighter with a higher $a - b$. Given the objective of minimizing $a - b$, the optimal solution thus implies $a^\star = 0$. $\square$

A.6 Proof of Proposition 5

The proof has two parts:

Proof. **Part (i).** Suppose the government chooses $(\tilde{a}, \tilde{b}) \in \mathbf{S}$ such that $(\tilde{a}, \tilde{b}) \neq (a^\star, b^\star)$. We can show that the utility is then lower. To see this, we need to show that

$$u\left(y + a - b + \left(\frac{\tilde{b} - \tilde{a}}{1+r}\right)\right) + \frac{\beta}{1-\beta} u\left(y + (1-\beta)(\tilde{a} - \tilde{b})\right)$$
$$< u\left(y + a - b + \left(\frac{b^\star - a^\star}{1+r}\right)\right) + \frac{\beta}{1-\beta} u\left(y + (1-\beta)(a^\star - b^\star)\right). \quad \text{(A.19)}$$

Denote $\Delta \equiv \frac{b^\star - a^\star}{1+r} - \frac{\tilde{b} - \tilde{a}}{1+r}$, $c^\star \equiv y + a - b + \left(\frac{b^\star - a^\star}{1+r}\right)$, and $\hat{c} \equiv y + (1-\beta)(a^\star - b^\star)$. We know that by definition of $(a^\star, b^\star)$, $\Delta > 0$. Using these expressions, we can rewrite (A.19) as

$$u(c^\star - \Delta) + \frac{\beta}{1-\beta} u\left(\hat{c} + \left(\frac{1-\beta}{\beta}\right)\Delta\right) < u(c^\star) + \frac{\beta}{1-\beta} u\left(\hat{c}\right).$$

Rearranging, we need to show that the following holds:

$$u(c^\star) - u(c^\star - \Delta) > \frac{\beta}{1-\beta}\left[u\left(\hat{c} + \left(\frac{1-\beta}{\beta}\right)\Delta\right) - u(\hat{c})\right]. \quad \text{(A.20)}$$

The result follows from an application of the mean-value theorem, the strict concavity of $u(\cdot)$, and the fact that $c^\star < \hat{c}$. Namely, there exists $x \in (c^\star - \Delta, c^\star)$, such that

$$u'(x)\Delta = u(c^\star) - u(c^\star - \Delta). \quad \text{(A.21)}$$

Similarly, there exists $z \in \left(\hat{c}, \hat{c} + \left(\frac{1-\beta}{\beta}\right)\Delta\right)$, such that

$$u'(z)\Delta = \frac{u\left(\hat{c} + \left(\frac{1-\beta}{\beta}\right)\Delta\right) - u(\hat{c})}{(1-\beta)/\beta}. \tag{A.22}$$

Since $z > x, \Delta > 0$ and $u(\cdot)$ is strictly concave, we have that $u'(x) > u'(z)$. Using this strict inequality and rearranging (A.21) and (A.22), we obtain (A.20), as we wanted to show.

□

Proof. **Part (ii).** Suppose, toward a contradiction, that $a' > 0$. Consider a portfolio such that

$$(\tilde{a}, \tilde{b}) = \left(a' - \Delta, b' - \frac{\Delta}{q(a', b')(1+r)}\right) \tag{A.23}$$

and the government exits in the same number of steps. By construction, the portfolio delivers the same level of consumption and utility at time t. Moreover, assume that tomorrow the government keeps the same $t+1$ policy (a'', b'') under repayment and sets $c_D = y + (1-\beta)(a' - \Delta)$ under default. This implies that the new portfolio delivers the same continuation value for $t+2$ in case of repayment.

We will show that the expected utility under the alternative policy is higher from $t+1$ onward, thus it is preferred to the policy with $a' > 0$. Let's define $\tilde{W}$ as follows

$$\begin{aligned}\tilde{W} = \frac{\lambda}{1-\beta} u\left(y + (1-\beta)(a' - \Delta)\right) + \\ (1-\lambda) u\Bigg(y + (a' - \Delta) - \frac{a''}{1+r} - \kappa\left(b' - \frac{\Delta}{q(a', b')(1+r)}\right) + \\ q(a'', b'')\left[b'' - (1-\delta)\left(b' - \frac{\Delta}{q(a', b')(1+r)}\right)\right]\Bigg).\end{aligned} \tag{A.24}$$

From the expression above we see that the first term of $\tilde{W}$ is the lifetime expected utility under default, and the second term is the $t+1$ expected utility.

Noticing from (4) that $q(a', b')(1+r) = (1-\lambda)\left[\kappa + (1-\delta)q(a'', b'')\right]$, and rearranging terms, we obtain

$$\begin{aligned}\tilde{W} = \frac{\lambda}{1-\beta} u\left(y + (1-\beta)(a' - \Delta)\right) + \\ (1-\lambda) u\left(y + a' - \frac{a''}{1+r} - \kappa b' + q(a'', b'')\left[b'' - (1-\delta)b'\right] + \Delta\frac{\lambda}{1-\lambda}\right).\end{aligned} \tag{A.25}$$

We want to show that

$$\tilde{W} > \frac{\lambda}{1-\beta} u\left(y + (1-\beta)a'\right) + (1-\lambda) u\left(y + a' - \frac{a''}{1+r} - \kappa b' + q(a'', b'')\left[b'' - (1-\delta)b'\right]\right),$$

where the right-hand side of the above inequality is an expression akin to $\tilde{W}$ but for the original portfolio with $a' > 0$. Denote $c_R^+(a', b') = y + a' - \frac{a''}{1+r} - \kappa b' + q(a'', b'')\left[b'' - (1-\delta)b'\right]$, and $c_D(a') = y + (1-\beta)a'$. Let's also define $\hat{\Delta} = \Delta\frac{\lambda}{1-\lambda}$. Plugging (A.25) into the last expression and using these expressions, we obtain that what we need to show is:

$$u\left(c_R^+(a', b') + \hat{\Delta}\right) - u\left(c_R^+(a', b')\right) > \frac{\lambda}{(1-\lambda)(1-\beta)}\left[u\left(c_D(a')\right) - u\left(c_D(a') - \frac{(1-\lambda)(1-\beta)}{\lambda}\hat{\Delta}\right)\right].$$

The result, as in the proof of part (i) of this proposition, follows from an application of the mean-value theorem. Namely, there exists $x \in (c_R^+(a', b'), c_R^+(a', b') + \hat{\Delta})$, such that

$$u'(x)\hat{\Delta} = u(c_R^+(a', b') + \hat{\Delta}) - u(c_R^+(a', b')). \tag{A.26}$$

Similarly, there exists $z \in \left(c_D(a') - \frac{(1-\lambda)(1-\beta)}{\lambda}\hat{\Delta}, c_D(a')\right)$, such that

$$u'(z)\hat{\Delta} = \frac{u\left(c_D(a')\right) - u\left(c_D(a') - \frac{(1-\lambda)(1-\beta)}{\lambda}\hat{\Delta}\right)}{\frac{(1-\lambda)(1-\beta)}{\lambda}}. \tag{A.27}$$

Since $c_D(a') > c_R^+(a', b')$ (from Lemma A.1), we can find a $\Delta > 0$ (and arbitrarily small) such that $z > x$. Then, from the strict concavity of the utility function, we have that $u'(x) > u'(z)$. Using this strict inequality and rearranging (A.26) and (A.27), we obtain the inequality we wanted to show.

□

A.7 Proof of Corollary 1

Proof. Part (ii) of Proposition 5 indicates that the government chooses $a_{t+1} = 0$ for all $t < T$ (i.e., as long as it remains in the crisis zone). Part (i) of Proposition 5 indicates that the government exits the crisis zone by choosing $(a^\star, b^\star)$. Once the government reaches the safe zone, it stays in the safe zone.

□

A.8 Proof of Proposition 6

Proof. If the government picks $(a^\star, b + a^\star - a)$, then the government is safe tomorrow. This follows from the fact that $b + a^\star - a < b^\star$ and that $b^-(a)$ is increasing in a, as shown in Proposition 2. In addition, notice that $c = y + (1 - \beta)(a - b)$ and the value of choosing this portfolio is given by $V_S(a - b)$. Given that $V_R^+(a, b) \leq V_S(a - b)$, it thus follows that the portfolio $(a^\star, b + a^\star - a)$, achieves the optimal solution.

□

B Additional Figures: Economy without Reserves

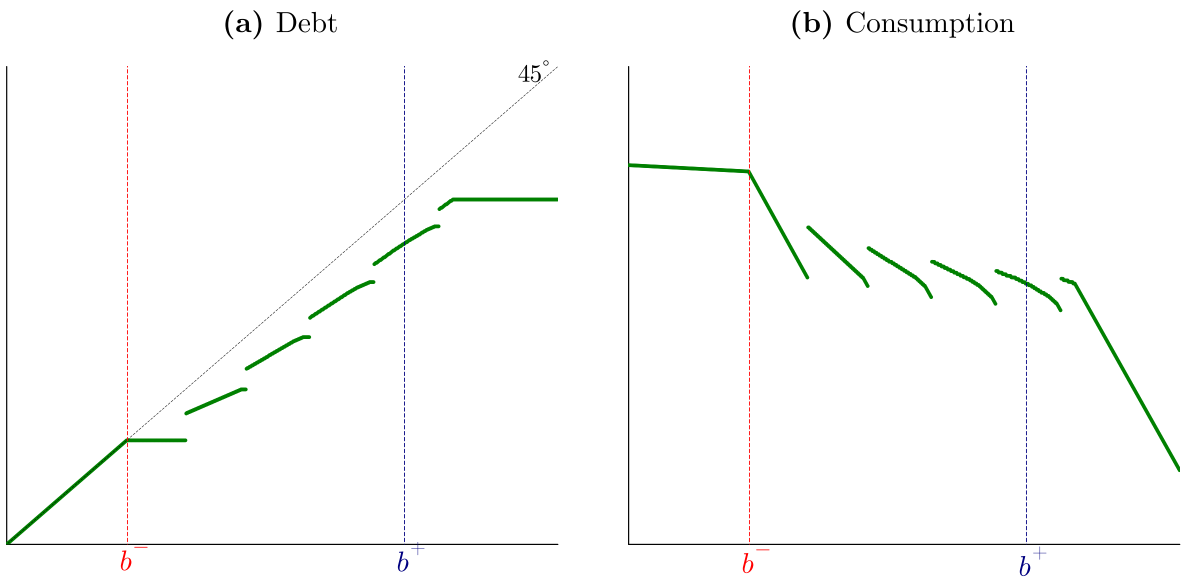

Figure A1: Policy without reserves. The figure presents the policy functions for the economy when the government is restricted from accumulating reserves. Panel (a) shows the policy for debt, $b'(b)$. Panel (b) exhibits the policy for consumption, $c(b)$.

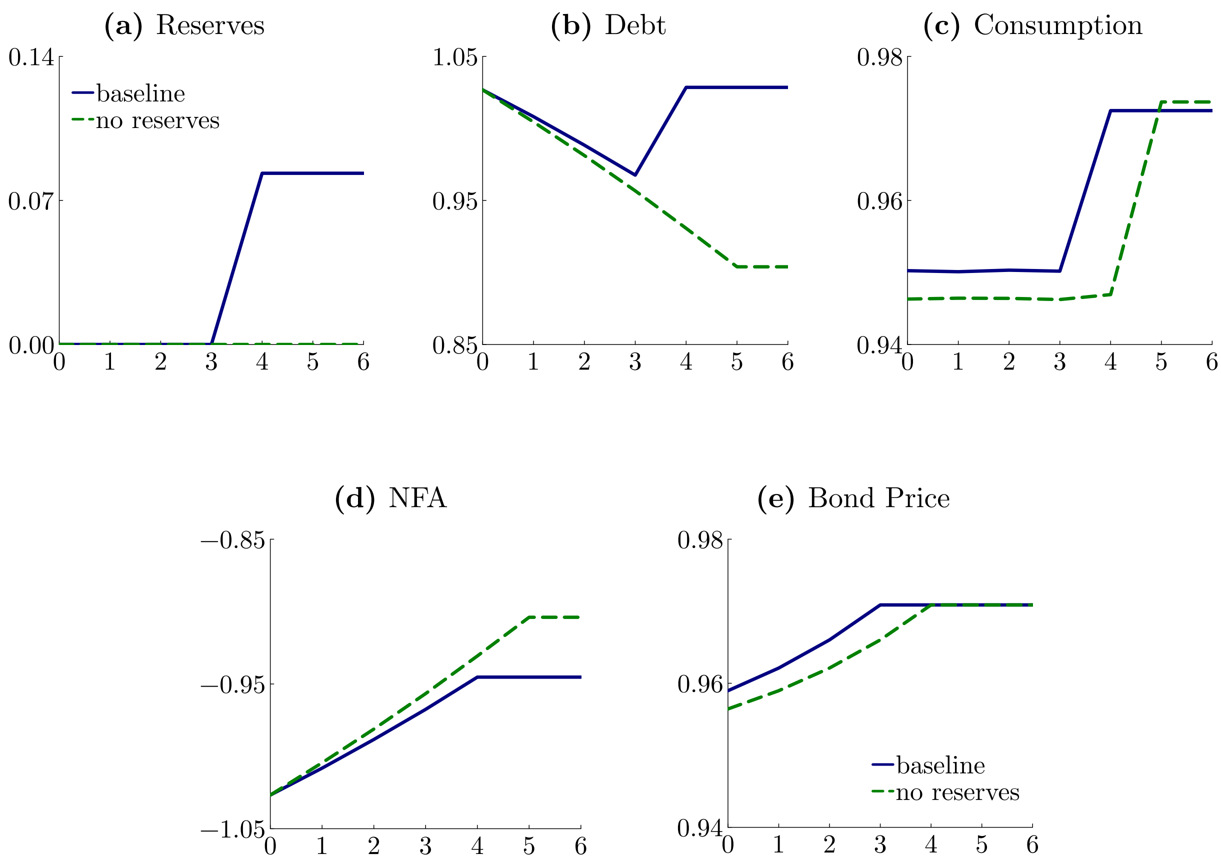

Figure A2: Lower consumption and longer to exit without reserves. The government is assumed to start in the crisis zone with $b = 1.04$ and zero initial reserves.

C Algorithm

1. Specify a grid for reserves and debt, which we denote respectively by A and B. We use 301 points in A and 1201 points in B.

2. For arbitrary value a, use $V_R(a, b^-(a)) = V_D(a)$ to find $b^-(a)$:

$$\max_{a' \geq 0} \left\{ u\left(y + a - \left(\frac{\delta + r}{1+r} \right) b^-(a) - \frac{a'}{1+r} \right) + \frac{\beta u(y + (1-\beta)(a' - (1-\delta) b^-(a))}{1-\beta} \right\} = \frac{u(y + (1-\beta)a) - \phi}{1-\beta},$$

employing a bisection scheme. Notice that this computation assumes that the government will be safe tomorrow after repaying the coupons due while facing a run today. We then verify that this is the case in equilibrium.[24]

3. Compute $(a^\star, b^\star)$.

4. Compute $V_R^-(a, b)$ for each grid point.

$$V_R^-(a, b) = \max_{a' \geq 0} \left\{ u\left(y + a - \left(\frac{\delta + r}{1+r} \right) b - \frac{a'}{1+r} \right) + \frac{\beta u(y + (1-\beta)(a' - (1-\delta) b)}{1-\beta} \right\}$$

5. Set an iterator counter $t = 1$.

6. For each $(a, b) \in A \times B$, start with an initial guess for $\tilde{q}_t(a, b) = \frac{1}{1+r}$ and $V_{R,t}^+(a, b) = \frac{u(y-(1-\beta)b_{\max})}{1-\beta}$, where $b_{\max}$ denotes the upper-bound of the debt grid.

7. For each $(a, b) \in A \times B$, go over all possible pairs $(\tilde{a}, \tilde{b}) \in A \times B$, searching for the optimal value under repayment with access to funding as follows:

   (a) If (a, b) satisfies $b \leq b^-(a)$, the government is safe under (a, b); i.e., $(a, b) \in \mathbf{S}$. Set $(a'_{t+1}, b'_{t+1}) = (a, b)$ and $V(a, b) = V_S(a - b)$.

   (b) If (a, b) satisfies $b > b^-(a)$, then evaluate the welfare of all candidate pairs $(\tilde{a}, \tilde{b}) \in A \times B$ and store the maximizing values (a'_{t+1}, b'_{t+1}); that is,

$$(a'_{t+1}, b'_{t+1}) \in \underset{(\tilde{a}, \tilde{b}) \in A \times B}{\operatorname{argmax}} W_{R,t}^+(a, b, \tilde{a}, \tilde{b}),$$

[24] It is also straightforward to extend the algorithm to consider parameterizations where the government can remain vulnerable tomorrow.

where

$$W_{R,t}^{+}(a,b,\tilde{a},\tilde{b}) \equiv u\left(y + a - \left(\frac{\delta + r}{1+r}\right) b - \frac{\tilde{a}}{1+r} + \tilde{q}_t(\tilde{a},\tilde{b})\left(\tilde{b} - (1-\delta)b\right)\right) + \beta \mathbb{E} V_t(\tilde{a},\tilde{b}),$$

where we compute the expected continuation according to equation (13).

8. Store the value $V_{R,t+1}^{+}(a,b)$ as

$$V_{R,t+1}^{+}(a,b) \equiv W_{R,t}^{+}(a,b,a'_{t+1},b'_{t+1}).$$

9. Update the threshold $b^{+}(a)$ using V_R^{+} computed above and V^D, which is given by (12).

10. Given the updated policy functions, compute an update for the bond price schedule for each $(a',b') \in A \times B$ as follows:

$$\tilde{q}_{t+1}(a',b') = \begin{cases} \frac{1}{1+r} & \text{if } (a',b') \in \mathbf{S}, \\ 0 & \text{if } (a',b') \in \mathbf{D}, \\ \frac{1-\lambda}{1+r}\left(\frac{\delta+r}{1+r} + (1-\delta)\tilde{q}_t(a'',b'')\right) & \text{if } (a',b') \in \mathbf{C}. \end{cases}$$

where (a'',b'') is the optimal portfolio under state (a',b'); i.e., $a'' \equiv a'_{t+1}(a',b')$, and $b'' \equiv b'_{t+1}(a',b')$.

11. Check for convergence. Compute

$$\begin{aligned} \epsilon_q &\equiv \|q_{t+1} - q_t\|_\infty, \\ \epsilon_{V_R^+} &\equiv \|V_{R,t+1}^{+} - V_{R,t}^{+}\|_\infty, \\ \epsilon_a &\equiv \|a'_{t+1} - a'_t\|_\infty, \\ \epsilon_b &\equiv \|b'_{t+1} - b'_t\|_\infty. \end{aligned}$$

If $\max\{\epsilon_q, \epsilon_{V_R^+}, \epsilon_a, \epsilon_b\} < 10^{-8}$, stop. Otherwise, set

$$\tilde{q}_t(a,b) = \xi \tilde{q}_{t+1}(a,b) + (1-\xi)\tilde{q}_t(a,b),$$

update the iterator counter to $t+1$ and go back to step (8). We set $\xi = 0.25$.